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Glasser-Manna integrals and the harmonic zeta function

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Abstract Using a new representation of the harmonic zeta function ζ_H , we establish a link between this function and a certain type of complex integral previously introduced by Glasser and Manna in relation with the study of the Laplace transform of the digamma function.

Keywords Harmonic numbers; Harmonic Stieltjes constants; Zeta values; Stieltjes constants; Bernoulli numbers; Digamma function; Glasser-Manna integrals.

1 Introduction

The harmonic zeta function ζ_H is defined for $\operatorname{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s},$$

where, for all integers $n \geq 1$,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Four decades ago, Apostol and Vu [2] and Matsuoka [15], completing original research of Euler [11], showed that this function could be continued as a meromorphic function with a double pole at $s = 1$, and simple poles at $s = 0$ and at odd negative integers. The special values of ζ_H at positive integers are given by Euler's formula [2, 11, 15]:

$$2\zeta_H(p) = (p+2)\zeta(p+1) - \sum_{k=1}^{p-2} \zeta(k+1)\zeta(p-k) \quad (p \geq 2),$$

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whereas the special values at even negative integers are given by Matsuoka's formula [5, Equation (16)], [15]:

$$2\zeta_H(-2p) = (1 - 2p)\zeta(1 - 2p) = (2p - 1)\frac{B_{2p}}{2p} \quad (p \geq 1),$$

where the B_{2p} are the Bernoulli numbers.

The Laurent expansion of the harmonic zeta function ζ_H around its double pole can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_{H,k} (s-1)^k \quad (0 < |s-1| < 1),$$

where $\gamma = -\Gamma'(1)$ is Euler's constant, and the coefficients $\gamma_{H,k}$ are called *harmonic Stieltjes constants* by analogy with the classical Stieltjes constants. A common definition of the classical Stieltjes constants γ_k for arbitrary k is

$$\gamma_k = \lim_{s \rightarrow 1} \left\{ (-1)^k \zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}} \right\} \quad (k \geq 0),$$

where $\zeta^{(k)}(s)$ is the k th derivative of the Riemann zeta function [3, 8]. In particular, γ_0 is Euler's constant γ . An explicit expression of the constant $\gamma_{H,0}$ is given by the following formula [6, Equation (6)]:

$$\gamma_{H,0} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\Gamma^{(2)}(1) = 0.989055\dots \quad (1)$$

This nice expression arises from a special case of a general formula which applies to height one multiple zeta functions $\zeta(s, 1, \dots, 1)$ of arbitrary depth [16, Equation (28)].

In the next section, we give a new representation of ζ_H (see Theorem 1) which is a subtle improvement of an earlier formula due to Boyadzhiev et al. [5, Theorem 1]. We use this representation to derive a new expression of the harmonic Stieltjes constant $\gamma_{H,k}$ (see Proposition 1) to be compared with the unwieldy one given by Kargin et al. [13, Equation (21)]. Furthermore, thanks to this representation of $\zeta_H(s)$, we also derive new integral representations of $\zeta(n)$ and $\zeta'(n)$ (see Corollary 1).

Another remarkable consequence of the representation of ζ_H given by Theorem 1 is that it allows us to establish a link between this function and a certain type of complex integral introduced earlier by Glasser and Manna [1, 10, 12] that proved to be useful in the study of the Laplace transform of the digamma function (see Definition 1 and Remark 5). To our knowledge, this unexpected connection between the special values of ζ_H and these integrals had never been noticed before.

2 New representation of $\zeta_H(s)$

In this section, we give a new expression of the harmonic zeta function which is a subtle improvement of [5, Theorem 1].

Theorem 1. *For all complex numbers s in $\mathbb{C} \setminus \mathbb{Z}$, we have*

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\Phi(s) \quad (2)$$

with

$$\Phi(s) := \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left(\text{Log}(1 + e^{ix}) \right)^{s-1} dx, \quad (3)$$

where Log denotes the principal value of the complex logarithm.

Proof. Let $z = \text{Log}(1 + e^{ix}) = \ln\left(2 \cos\left(\frac{x}{2}\right)\right) + \frac{ix}{2}$, $-\pi < x < \pi$. When x varies from $-\pi$ to π , the variable z travels the path L defined by the parametric equations $\text{Re } z = \ln(2 \cos(x/2))$ and $\text{Im } z = x/2$. This path extends from $-\infty$ below the line $\text{Im } z = 0$, passes through the point $(\ln 2, 0)$, then extends back to $-\infty$ above the line $\text{Im } z = 0$. The path L is homotopic to the Hankel contour used in [4] and [5]. Differentiation of the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z}{1-e^z} dz$$

leads to the following identity:

$$\psi(1-s)\zeta(s) + \zeta'(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1-e^z} dz \quad (s \neq 1, 2, 3, \dots),$$

where $\psi = \Gamma'/\Gamma$ is the digamma function [4, Equation (2.7)]. Writing

$$\frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1-e^z} dz = \phi(s) - \Phi(s),$$

with

$$\phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}\left(\frac{e^z - 1}{z}\right) dz$$

and

$$\Phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}(e^z - 1) dz,$$

we deduce the following identity:

$$\Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s) = \Gamma(1-s)\Phi(s). \quad (4)$$

By means of the relation [5, Equation (25)]:

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s)$$

we then derive formula (2) from (4) with the expression of $\Phi(s)$ given by formula (3). \square

3 Some remarkable consequences

3.1 New expression of $\gamma_{H,k}$

We can deduce from Theorem 1 an expression of $\gamma_{H,k}$ in terms of integrals $L_k = \Phi^{(k)}(1)$. More precisely, we have the following result:

Proposition 1. For any integer $k \geq 0$, if L_k , ξ_k and ω_k are respectively defined by

$$\begin{aligned} L_k &:= \Phi^{(k)}(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \operatorname{Log}^k(\operatorname{Log}(1 + e^{ix})) dx, \\ \xi_k &:= (-1)^k \Gamma^{(k)}(1) = (-1)^k \int_0^{+\infty} e^{-x} \ln^k(x) dx, \\ \omega_k &:= (-1)^k \zeta^{(k)}(2) = \sum_{n=1}^{\infty} \frac{\ln^k n}{n^2}, \end{aligned}$$

then we have

$$L_1 = \Phi'(1) = -\frac{3}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1 = -2.561174\dots \quad (5)$$

and, for each positive integer n , the following general relations hold true:

$$\begin{aligned} \gamma_{H,2n-1} = \omega_{2n-1} - \frac{\gamma_{2n}}{2n} + \frac{1}{2n} \sum_{k=0}^{2n-1} \binom{2n}{k} \xi_k L_{2n-k} \\ + \sum_{k=1}^n \frac{2(2n-1)!}{(2n-2k)!} \gamma_{2n-2k} \zeta(2k), \quad (6) \end{aligned}$$

and

$$\begin{aligned} \gamma_{H,2n} = \omega_{2n} - \frac{\gamma_{2n+1}}{2n+1} - \frac{1}{2n+1} \sum_{k=0}^{2n} \binom{2n+1}{k} \xi_k L_{2n+1-k} \\ + \sum_{k=1}^n \frac{2(2n)!}{(2n+1-2k)!} \gamma_{2n+1-2k} \zeta(2k) - 2(2n)! \zeta(2n+2). \quad (7) \end{aligned}$$

Example 1. Applying formulas (6) and (7) to the simplest case $n = 1$ and using (5), we then obtain

$$\gamma_{H,1} = \omega_1 - \frac{1}{2}\gamma^3 - \frac{1}{2}\gamma_2 + \frac{1}{2}L_2 + \frac{1}{2}\gamma\zeta(2) - \gamma\gamma_1,$$

and

$$\gamma_{H,2} = \omega_2 + \frac{1}{2}\gamma^4 - \frac{1}{3}\gamma_3 - \frac{1}{3}L_3 - \gamma L_2 + (2\gamma^2 + 5\gamma_1)\zeta(2) + \gamma^2\gamma_1 - \frac{1}{4}\zeta(4).$$

Numerical evaluations of L_2 and L_3 are

$$L_2 = -1.924491\dots, \quad \text{and} \quad L_3 = 7.158075\dots$$

This leads to the following numerical evaluations:

$$\gamma_{H,1} = 0.400761\dots, \quad \text{and} \quad \gamma_{H,2} = 0.971304\dots$$

Proof of Proposition 1. The key formula to derive the relations (5)–(7) is the splitting of $\zeta_H(s)$ given by formula (2). Fortunately, the Laurent series expansion of each component in (2) can be written explicitly.

a) The Laurent expansions of $\pi \cot(\pi s)$ and $\zeta(s)$ at $s = 1$ are respectively

$$\pi \cot(\pi s) = \frac{1}{s-1} - \sum_{k=1}^{\infty} 2\zeta(2k)(s-1)^{2k-1},$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{\gamma_k}{k!} (s-1)^k,$$

where the coefficients γ_k are the Stieltjes constants. The expansion of $\pi \cot(\pi s) \zeta(s)$ is then obtained by Cauchy product:

$$\begin{aligned} \pi \cot(\pi s) \zeta(s) &= \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 - 2\zeta(2) - \left(-\frac{1}{2}\gamma_2 + 2\gamma\zeta(2)\right)(s-1) \\ &\quad + \frac{1}{2} \left(-\frac{1}{3}\gamma_3 + 4\zeta(2)\gamma_1 - 4\zeta(4)\right)(s-1)^2 \\ &\quad - \frac{1}{6} \left(-\frac{1}{4}\gamma_4 + 6\zeta(2)\gamma_2 + 12\gamma\zeta(4)\right)(s-1)^3 + \dots \quad (8) \end{aligned}$$

b) It follows from the definition of ξ_k that the Laurent expansion of $\Gamma(1-s)$ at $s = 1$ is given by

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \sum_{k=2}^{\infty} \frac{\xi_k}{k!} (s-1)^{k-1} \quad (|s-1| < 1),$$

On the other hand, the function Φ defined by (3) is an entire function of s with $\Phi(1) = L_0 = 0$, and the definition of L_k as $\Phi^{(k)}(1)$ implies that the Taylor expansion of $\Phi(s)$ at $s = 1$ is given by

$$\Phi(s) = \sum_{k=1}^{\infty} \frac{L_k}{k!} (s-1)^k.$$

The Laurent expansion of $\Gamma(1-s)\Phi(s)$ then follows by Cauchy product:

$$\begin{aligned} \Gamma(1-s)\Phi(s) &= -L_1 - \left(\frac{1}{2}L_2 + \gamma L_1\right)(s-1) \\ &\quad + \frac{1}{2}\left(-\frac{1}{3}L_3 - \gamma L_2 - \xi_2 L_1\right)(s-1)^2 \\ &\quad - \frac{1}{6}\left(\frac{1}{4}L_4 + \gamma L_3 + \frac{3}{2}\xi_2 L_2 + (2\zeta(3) + 3\gamma\zeta(2) + \gamma^3)L_1\right)(s-1)^3 + \dots \end{aligned} \quad (9)$$

c) The Taylor expansion of $\zeta(s+1)$ at $s=1$ can be written as follows:

$$\zeta(s+1) = \zeta(2) + \sum_{k=1}^{\infty} (-1)^k \frac{\omega_k}{k!} (s-1)^k. \quad (10)$$

By assembling equations (8)–(10) above, we first obtain (by identifying the constant term) the relation $\gamma_{H,0} = -\gamma_1 - \zeta(2) - L_1$ which, thanks to (1), gives (5). In the same way, the general formulae (6) and (7) are easily derived by identifying the coefficients of higher degree. \square

Remark 1. For $2 \leq k \leq n$, the constants ξ_n involved in formulae (6)–(7) have a polynomial expression in terms of Euler's constant γ and $\zeta(k)$ [3, 7]. More precisely, we have

$$\xi_n = n! P_n(\gamma, \zeta(2), \dots, \zeta(n)) \quad (n \geq 2),$$

where, for arbitrary n , the polynomials P_n are defined by the generating function

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, \dots, x_n) t^n = \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + x_1 t + \frac{1}{2}(x_1^2 + x_2) t^2 + \dots$$

In particular, $\xi_1 = \gamma$, and

$$\xi_2 = \gamma^2 + \zeta(2), \quad \xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3), \quad \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3(\zeta(2))^2 + 6\zeta(4).$$

Remark 2. Using the notations of [16], if $\gamma_k^{[2]}$ denotes the k th Stieltjes constant of the height one double zeta function $\zeta(s, 1) = \sum_{n>m>0} \frac{1}{n^s m}$ then, in view of the relation $\zeta(s, 1) + \zeta(s+1) = \zeta_H(s)$, we have the simple relation [16, Equation (14)]:

$$\gamma_k^{[2]} + \omega_k = \gamma_{H,k} \quad (k \geq 0).$$

Hence formulas (6) and (7) also apply to $\gamma_k^{[2]}$ with only a slight modification. Moreover, from a theorem of Ramanujan [8, Theorem 4], we have the identity

$$\omega_k = k! + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_{k+n} \quad (k \geq 1).$$

Remark 3. The constants ξ_n involved in formulae (6)–(7) have a polynomial expression in terms of Euler’s constant γ and $\zeta(k)$ for $2 \leq k \leq n$ [3, 7]. More precisely,

$$\xi_n = n! P_n(\gamma, \zeta(2), \dots, \zeta(n)) \quad (n \geq 2),$$

where, for arbitrary n , the polynomials P_n are defined by the generating function

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, \dots, x_n) t^n = \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right).$$

In particular, we have

$$\xi_2 = \gamma^2 + \zeta(2), \quad \xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3), \quad \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3(\zeta(2))^2 + 6\zeta(4).$$

3.2 New integral representations of $\zeta(n)$ and $\zeta'(n)$

Applied in a neighborhood of $s = n$ (for an integer $n \geq 2$), formula (2) provides a relation between $\Phi(s)$ and $\zeta_H(s)$ which allows us to derive nice evaluations of $\Phi(n)$ and $\Phi'(n)$.

Proposition 2. For any integer $n \geq 2$, we have

$$\Phi(n) = (-1)^{n-1} (n-1)! \zeta(n), \quad (11)$$

and

$$\begin{aligned} \Phi'(n) &= (-1)^{n-1} (n-1)! \\ &\times \left\{ \zeta'(n) + \zeta(n) \psi(n) - \frac{1}{2} n \zeta(n+1) + \frac{1}{2} \sum_{k=1}^{n-2} \zeta(n-k) \zeta(k+1) \right\}. \quad (12) \end{aligned}$$

Proof of Proposition 2. For $n \geq 2$, we can write

$$\begin{aligned} \pi \cot(\pi s) \zeta(s) &= \frac{\zeta(n)}{s-n} + \zeta'(n) + O(s-n), \\ \Gamma(1-s) &= \frac{(-1)^n}{(n-1)!} \left(\frac{1}{s-n} + (\gamma - H_{n-1}) \right) + O(s-n), \end{aligned}$$

and

$$\Gamma(1-s) \Phi(s) = \frac{(-1)^n}{(n-1)!} \frac{\Phi(n)}{s-n} + \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + O(s-n).$$

By applying formula (2) around $s = n$, we obtain the equation

$$\begin{aligned} \zeta_H(n) + O(s - n) &= \left(\zeta(n) + \frac{(-1)^n}{(n-1)!} \Phi(n) \right) \frac{1}{s-n} \\ &+ \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + \zeta'(n) + \zeta(n+1) + O(s-n), \end{aligned}$$

which allows us to derive the identities $\zeta(n) = \frac{(-1)^{n+1}}{(n-1)!} \Phi(n)$, and

$$\zeta_H(n) = \frac{(-1)^n}{(n-1)!} (\Phi'(n) - \Phi(n)\psi(n)) + \zeta'(n) + \zeta(n+1).$$

Formulae (11) and (12) then follow using Euler's formula for $\zeta_H(n)$. \square

We deduce from Proposition 2, the following corollary:

Corollary 1. For any integer $n \geq 2$, we have

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left(\text{Log}(1 + e^{ix}) \right)^{n-1} dx. \quad (13)$$

and

$$\begin{aligned} \zeta'(n) &= \frac{1}{2} n \zeta(n+1) - \zeta(n) \psi(n) - \frac{1}{2} \sum_{k=1}^{n-2} \zeta(n-k) \zeta(k+1) \\ &+ \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log} \left(\text{Log}(1 + e^{ix}) \right) \left(\text{Log}(1 + e^{ix}) \right)^{n-1} dx. \quad (14) \end{aligned}$$

Remark 4. It should be noted that our formula (13) for $\zeta(n)$ is similar to but simpler than the one given by Boyadzhiev [4, Equation (2.6)] which, after the substitution $z = \text{Log}(1 + e^{ix})$, translates into

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Log} \left(\text{Log}(1 + e^{ix}) \right) \left(\text{Log}(1 + e^{ix}) \right)^{n-1} dx.$$

Example 2. Applying the previous formulas to the simplest cases $n = 2$ and $n = 3$, we get

$$\begin{aligned} \zeta(2) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log}(1 + e^{ix}) dx, \\ \zeta(3) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} ix \text{Log}^2(1 + e^{ix}) dx, \\ \zeta'(2) &= \zeta(3) - \zeta(2)(1 - \gamma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log} \left(\text{Log}(1 + e^{ix}) \right) \text{Log}(1 + e^{ix}) dx. \end{aligned}$$

3.3 Link with the Glasser-Manna integral

Definition 1. The Glasser-Manna function M [1, 10, 12] is defined by

$$M(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(e^{-z}(1+e^{ix}))} dx = \frac{4}{\pi} \int_0^{\pi/2} \frac{y^2}{y^2 + \ln^2(2e^{-z} \cos(y))} dy.$$

It is analytic for z in the disc $D(0, \ln 2)$ and verifies the relation

$$M^{(n)}(0) = n! \Phi(-n) \quad (n \geq 0). \quad (15)$$

Remark 5. Let $\mathcal{L}\psi(x+1)$ be the Laplace transform of the function $x \mapsto \psi(x+1)$, where ψ is the digamma function, the following relation can be shown:

$$M(z) = \frac{\gamma}{z} + \mathcal{L}\psi(x+1)(z) - \frac{1}{e^{-z} - 1} (z + \operatorname{Log}(1 - e^{-z})) \quad (0 < |z| < \ln(2)).$$

Applied in a neighborhood of $s = -n$, formula (2) provides a relation between $\Phi(s)$ and $\zeta_H(s)$ which allows us to give a nice evaluation of $\Phi(-n)$ for any non-negative integer n .

Proposition 3. We have

$$2\Phi(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1+e^{ix})} dx = \ln(2\pi) - \gamma + 1, \quad (16)$$

and

$$\begin{aligned} 2\Phi'(0) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix \operatorname{Log}(\operatorname{Log}(1+e^{ix}))}{\operatorname{Log}(1+e^{ix})} dx \\ &= \gamma_1 + \frac{1}{2} \ln^2(2\pi) + \frac{1}{2} \gamma^2 - \gamma - \gamma \ln(2\pi) - \frac{7}{4} \zeta(2) + 2\beta, \end{aligned} \quad (17)$$

where β is the linear coefficient in the Laurent expansion of ζ_H at $s = 0$.

Proof of Proposition 3. Around 0, we have the decomposition given by (2):

$$\zeta_H(s) = \zeta(s+1) + \pi \cot(\pi s) \zeta(s) + \Gamma(1-s)\Phi(s),$$

and the expansion [6, Equation (8)]

$$\zeta_H(s) = \frac{1}{2s} + \frac{1}{2}\gamma + \frac{1}{2} + \beta s + O(s^2).$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s+1) &= \frac{1}{s} + \gamma - \gamma_1 s + O(s^2), \\ \pi \cot(\pi s) \zeta(s) &= -\frac{1}{2s} - \frac{1}{2} \ln(2\pi) + \left(\frac{1}{2}\gamma_1 - \frac{1}{4} \ln^2(2\pi) + \frac{1}{4}\gamma^2 + \frac{7}{8}\zeta(2)\right)s + O(s^2), \\ \Gamma(1-s)\Phi(s) &= \Phi(0) + (\Phi'(0) + \gamma\Phi(0)) + O(s^2). \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand side of (2), we deduce the equation

$$\frac{1}{2}\gamma + \frac{1}{2} = \gamma - \frac{1}{2}\ln(2\pi) + \Phi(0),$$

which is equivalent to (16). In the same way, formula (17) is derived by identifying the linear coefficient in the expansion of the right-hand side of (2). \square

Remark 6. The constant β which occurs in formula (17) can be evaluated using [16, Corollary 4.2]; more precisely, we have

$$\beta = 1 + \frac{1}{2}\gamma - \frac{1}{4}\gamma^2 - \gamma_1 + \frac{1}{4}\zeta(2) - \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^2} = 1.589935\dots,$$

where the b_n are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n x^n \quad (|x| < 1).$$

Proposition 4. We have

$$\Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^2(1+e^{ix})} dx = \ln(\mathcal{A}) - \frac{1}{12}\gamma + \frac{7}{24}, \quad (18)$$

where $\ln(\mathcal{A}) = \frac{1}{12} - \zeta'(-1)$ and, more generally, for all integers $n \geq 2$, we have

$$(2n-1)! \Phi(1-2n) = \ln(\mathcal{A}_n) - \frac{B_{2n}}{2n} \gamma + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2}, \quad (19)$$

with $\ln(\mathcal{A}_n) := \frac{H_{2n-1} B_{2n}}{2n} - \zeta'(1-2n)$. The numbers \mathcal{A}_n for $n \geq 2$ are called *generalized Glaisher-Kinkelin constants* (see [14] for more details).

Example 3. In particular,

$$\Phi(-3) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^4(1+e^{ix})} dx = \frac{1}{6} \ln(\mathcal{A}_2) + \frac{1}{720}\gamma + \frac{1}{320}.$$

Proof of Proposition 4. Around $s = -1$, we have the expansion [6, Equation (9)]

$$\zeta_H(s) = -\frac{1}{12(s+1)} - \frac{1}{12}\gamma - \frac{1}{8} + O(s+1).$$

On the other side, we have

$$\begin{aligned}\zeta(s+1) &= -\frac{1}{2} + O(s+1), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-1) + O(s+1), \\ \Gamma(1-s)\Phi(s) &= \Phi(-1) + O(s+1).\end{aligned}$$

Formula (18) is then deduced by identifying the constant coefficient in the expansion of the right-hand member of (2). In the same way, from [6, Proposition 2], the Laurent expansion around $s = 1 - 2n$ for $n \geq 2$ is given by

$$\zeta_H(s) = \frac{\zeta(1-2n)}{s+2n-1} - \frac{B_{2n}}{2n} \gamma + \frac{H_{2n-1} B_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} + O(s+2n-1)$$

which, by the same method, allows us to deduce formula (19). \square

Proposition 5. For all positive integers n , we have

$$(2n)! \Phi(-2n) = -\zeta'(-2n) + (2n+1) \frac{B_{2n}}{4n}, \quad (20)$$

from which follows the reflection formula:

$$2(2n)! \Phi(-2n) = (-1)^{n+1} (2\pi)^{-2n} \Phi(2n+1) + (2n+1) \frac{B_{2n}}{2n}. \quad (21)$$

Example 4. In particular,

$$2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^3(1+e^{ix})} dx = -\zeta'(-2n) + \frac{1}{8} = \frac{1}{4\pi^2} \zeta(3) + \frac{1}{8}.$$

Proof of Proposition 5. Around $s = -2n$, we have by Matsuoka's formula

$$\zeta_H(s) = (2n-1) \frac{B_{2n}}{4n} + O(s+2n).$$

On the other side, we have the expansions

$$\begin{aligned}\zeta(s+1) &= -\frac{B_{2n}}{2n} + O(s+2n), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-2n) + O(s+2n), \\ \Gamma(1-s)\Phi(s) &= (2n)! \Phi(-2n) + O(s+2n).\end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (2), we obtain formula (20). Moreover, the well-known identity

$$-2\zeta'(-2n) = (-1)^{n+1} (2\pi)^{-2n} (2n)! \zeta(2n+1)$$

and the relation $(2n)! \zeta(2n+1) = \Phi(2n+1)$ given by (11) enables to deduce formula (21) from (20). \square

4 Appendix: Evaluation of Glasser-Manna integrals by means of shifted Mascheroni series

In this appendix, we reinterpret some of the previous formulas in terms of shifted Mascheroni series. For any positive integer k , we consider the series

$$\sigma_k := \sum_{n=k+1}^{\infty} \frac{|b_n|}{n-k},$$

where the b_n are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \dots$$

These series σ_k , called *shifted Mascheroni series*, have been studied in detail in [9]. The nice identity

$$\sigma_1 = \frac{1}{2} \ln(2\pi) - \frac{1}{2} \gamma - \frac{1}{2}$$

is well-known [9, Proposition 2] and allows us to deduce from formula (16) an evaluation of $\Phi(0)$ in terms of σ_1 :

$$M(0) = \Phi(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1 + e^{ix})} dx = \sigma_1 + 1 = 1.13033\dots$$

This evaluation was first given by Glasser and Manna [12, Proposition 3.3] in a slightly different but equivalent form. More generally, the following formula:

$$\sum_{k=1}^n (-1)^{n+k} k! S(n, k) \sigma_{k+1} = -\zeta'(-n) - \frac{B_{n+1}}{n+1} \left(\gamma + \frac{1}{n+1} \right) \quad (n \geq 1),$$

with $S(n, k)$ denoting the Stirling numbers of the second kind [9, Proposition 3], allows us to give reinterpretations of formulas (18), (19) and (20) in terms of shifted Mascheroni series σ_k . In this way, we obtain

$$\begin{aligned} (2n-1)! \Phi(1-2n) &= M^{(2n-1)}(0) \\ &= \sum_{k=1}^{2n-1} (-1)^{k+1} k! S(2n-1, k) \sigma_{k+1} + \frac{B_{2n} H_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} \end{aligned} \quad (n \geq 1),$$

and

$$(2n)! \Phi(-2n) = M^{(2n)}(0) = \sum_{k=1}^{2n} (-1)^k k! S(2n, k) \sigma_{k+1} + (2n+1) \frac{B_{2n}}{4n} \quad (n \geq 1).$$

In particular, we have

$$M'(0) = \Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^2(1 + e^{ix})} dx = \sigma_2 + \frac{5}{12} = 0.49232\dots$$

and

$$M''(0) = 2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^3(1 + e^{ix})} dx = 2\sigma_3 - \sigma_2 + \frac{1}{8} = 0.15544\dots$$

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