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# Glasser-Manna integrals and the harmonic zeta function

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**Abstract** Using a new representation of the harmonic zeta function, we establish a link between the harmonic zeta function and a certain type of complex integral previously introduced by Glasser and Manna in relation with the study of the Laplace transform of the digamma function.

**Keywords** Harmonic numbers; Harmonic Stieltjes constants; Zeta values; Stieltjes constants; Bernoulli numbers; Digamma function; Glasser-Manna integrals.

## 1 Introduction

The harmonic zeta function  $\zeta_H$  is defined for  $\operatorname{Re}(s) > 1$  by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s},$$

where, for all integers  $n \geq 1$ ,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Four decades ago, Apostol and Vu [1] and Matsuoka [14], completing original research of Euler [10], showed that this function could be continued as a meromorphic function with a double pole at  $s = 1$ , and simple poles at  $s = 0$  and at odd negative integers. The special values of  $\zeta_H$  at positive integers are given by Euler's formula [1, 10, 14]:

$$2\zeta_H(p) = (p+2)\zeta(p+1) - \sum_{k=1}^{p-2} \zeta(k+1)\zeta(p-k) \quad (p \geq 2),$$

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whereas the special values at even negative integers are given by Matsuoka's formula [4, Equation (16)], [14]:

$$2\zeta_H(-2p) = (1 - 2p)\zeta(1 - 2p) = (2p - 1)\frac{B_{2p}}{2p} \quad (p \geq 1),$$

where the  $B_{2p}$  are the Bernoulli numbers.

The Laurent expansion of the harmonic zeta function  $\zeta_H$  around its double pole can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_{H,k} (s-1)^k \quad (0 < |s-1| < 1),$$

where  $\gamma = -\Gamma'(1)$  is Euler's constant, and the coefficients  $\gamma_{H,k}$  are called *harmonic Stieltjes constants* by analogy with the classical Stieltjes constants. A common definition of the classical Stieltjes constants  $\gamma_k$  for arbitrary  $k$  is

$$\gamma_k = \lim_{s \rightarrow 1} \left\{ (-1)^k \zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}} \right\} \quad (k \geq 0),$$

where  $\zeta^{(k)}(s)$  is the  $k$ th derivative of the Riemann zeta function [2, 7]. In particular,  $\gamma_0$  is Euler's constant  $\gamma$ . An explicit expression of the constant  $\gamma_{H,0}$  is given by the following formula [5, Equation (6)]:

$$\gamma_{H,0} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\Gamma^{(2)}(1) = 0.989055\dots \quad (1)$$

This nice expression arises from a special case of a general formula which applies to height one multiple zeta functions  $\zeta(s, 1, \dots, 1)$  of arbitrary depth [15, Equation (28)].

In the next section, we give a new representation of  $\zeta_H$  (see Theorem 1) which is an improvement of an earlier formula due to Boyadzhiev et al. [4, Theorem 1]. We use this representation to derive a new expression of the harmonic Stieltjes constant  $\gamma_{H,k}$  (see Proposition 1) to be compared with the unwieldy one given by Kargin et al. [12, Equation (21)]. Furthermore, thanks to this representation of  $\zeta_H(s)$ , we also derive new integral representations of  $\zeta(n)$  and  $\zeta'(n)$  (see Corollary 1).

Another remarkable consequence of the representation of  $\zeta_H$  given by Theorem 1 is that it allows us to establish a link with a certain type of complex integral introduced earlier by Glasser and Manna [9, 11] (see Definition 1) that proved to be useful in the study of the Laplace transform of the digamma function (see Remark 4). To our knowledge, this unexpected connection between the special values of  $\zeta_H$  and these integrals had never been noticed before.

## 2 New representation of $\zeta_H(s)$

In this section, we give a new expression of the harmonic zeta function which is a subtle improvement of [4, Theorem 1].

**Theorem 1.** *For all complex numbers  $s$  in  $\mathbb{C} \setminus \mathbb{Z}$ , we have*

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\Phi(s) \quad (2)$$

with

$$\Phi(s) := \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left( \text{Log}(1 + e^{ix}) \right)^{s-1} dx, \quad (3)$$

where  $\text{Log}$  denotes the principal value of the complex logarithm.

*Proof.* Let  $z = \text{Log}(1 + e^{ix}) = \ln\left(2 \cos\left(\frac{x}{2}\right)\right) + \frac{ix}{2}$ ,  $-\pi < x < \pi$ . When  $x$  varies from  $-\pi$  to  $\pi$ , the variable  $z$  travels the path  $L$  defined by the parametric equations  $\text{Re } z = \ln(2 \cos(x/2))$  and  $\text{Im } z = x/2$ . This path extends from  $-\infty$  below the line  $\text{Im } z = 0$ , passes through the point  $(\ln 2, 0)$ , then extends back to  $-\infty$  above the line  $\text{Im } z = 0$ . The path  $L$  is homotopic to the Hankel contour used in [3] and [4]. Differentiation of the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z}{1-e^z} dz$$

leads to the following identity:

$$\psi(1-s)\zeta(s) + \zeta'(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1-e^z} dz \quad (s \neq 1, 2, 3, \dots),$$

where  $\psi = \Gamma'/\Gamma$  is the digamma function [3, Equation (2.7)]. Writing

$$\frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1-e^z} dz = \phi(s) - \Phi(s),$$

with

$$\phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}\left(\frac{e^z - 1}{z}\right) dz$$

and

$$\Phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}(e^z - 1) dz,$$

allows us to rewrite the above identity as follows:

$$\Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s) = \Gamma(1-s)\Phi(s). \quad (4)$$

Using the relation

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s)$$

given by [4, Equation (25)], we then derive formula (2) from (4) with  $\Phi(s)$  given by formula (3).  $\square$

### 3 Some remarkable consequences

#### 3.1 New expression of $\gamma_{H,k}$

We can deduce from Theorem 1 an expression of  $\gamma_{H,k}$  in terms of integrals  $L_k = \Phi^{(k)}(1)$ . More precisely, we have the following result:

**Proposition 1.** For any integer  $k \geq 0$ , if  $L_k$ ,  $\xi_k$  and  $\omega_k$  are respectively defined by

$$\begin{aligned} L_k &:= \Phi^{(k)}(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \operatorname{Log}^k(\operatorname{Log}(1 + e^{ix})) dx, \\ \xi_k &:= (-1)^k \Gamma^{(k)}(1) = (-1)^k \int_0^{+\infty} e^{-x} \ln^k(x) dx, \\ \omega_k &:= (-1)^k \zeta^{(k)}(2) = \sum_{n=1}^{\infty} \frac{\ln^k n}{n^2}, \end{aligned}$$

then we have

$$L_1 = \Phi'(1) = -\frac{3}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1 = -2.561174\dots \quad (5)$$

and, for each positive integer  $n$ , the following general relations hold true:

$$\begin{aligned} \gamma_{H,2n-1} = \omega_{2n-1} - \frac{\gamma_{2n}}{2n} + \frac{1}{2n} \sum_{k=0}^{2n-1} \binom{2n}{k} \xi_k L_{2n-k} \\ + \sum_{k=1}^n \frac{2(2n-1)!}{(2n-2k)!} \gamma_{2n-2k} \zeta(2k), \quad (6) \end{aligned}$$

and

$$\begin{aligned} \gamma_{H,2n} = \omega_{2n} - \frac{\gamma_{2n+1}}{2n+1} - \frac{1}{2n+1} \sum_{k=0}^{2n} \binom{2n+1}{k} \xi_k L_{2n+1-k} \\ + \sum_{k=1}^n \frac{2(2n)!}{(2n+1-2k)!} \gamma_{2n+1-2k} \zeta(2k) - 2(2n)! \zeta(2n+2). \quad (7) \end{aligned}$$

**Example 1.** Applying formulas (6) and (7) to the simplest case  $n = 1$  and using (5), we then obtain

$$\gamma_{H,1} = \omega_1 - \frac{1}{2}\gamma^3 - \frac{1}{2}\gamma_2 + \frac{1}{2}L_2 + \frac{1}{2}\gamma\zeta(2) - \gamma\gamma_1,$$

and

$$\gamma_{H,2} = \omega_2 + \frac{1}{2}\gamma^4 - \frac{1}{3}\gamma_3 - \frac{1}{3}L_3 - \gamma L_2 + (2\gamma^2 + 5\gamma_1)\zeta(2) + \gamma^2\gamma_1 - \frac{1}{4}\zeta(4).$$

Numerical evaluations of  $L_2$  and  $L_3$  are

$$L_2 = -1.924491\dots, \quad \text{and} \quad L_3 = 7.158075\dots$$

This leads to the following numerical evaluations:

$$\gamma_{H,1} = 0.400761\dots, \quad \text{and} \quad \gamma_{H,2} = 0.971304\dots$$

*Proof of Proposition 1.* The key formula to derive the relations (5)–(7) is the splitting of  $\zeta_H(s)$  given by formula (2). Fortunately, the Laurent series expansion of each component in (2) can be written explicitly.

a) The Laurent expansions of  $\pi \cot(\pi s)$  and  $\zeta(s)$  at  $s = 1$  are respectively

$$\pi \cot(\pi s) = \frac{1}{s-1} - \sum_{k=1}^{\infty} 2\zeta(2k)(s-1)^{2k-1},$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{\gamma_k}{k!} (s-1)^k,$$

where the coefficients  $\gamma_k$  are the Stieltjes constants. The expansion of  $\pi \cot(\pi s) \zeta(s)$  is then obtained by Cauchy product:

$$\begin{aligned} \pi \cot(\pi s) \zeta(s) &= \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 - 2\zeta(2) - \left(-\frac{1}{2}\gamma_2 + 2\gamma\zeta(2)\right)(s-1) \\ &\quad + \frac{1}{2} \left(-\frac{1}{3}\gamma_3 + 4\zeta(2)\gamma_1 - 4\zeta(4)\right)(s-1)^2 \\ &\quad - \frac{1}{6} \left(-\frac{1}{4}\gamma_4 + 6\zeta(2)\gamma_2 + 12\gamma\zeta(4)\right)(s-1)^3 + \dots \quad (8) \end{aligned}$$

b) It follows from the definition of  $\xi_k$  that the Laurent expansion of  $\Gamma(1-s)$  at  $s = 1$  is given by

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \sum_{k=2}^{\infty} \frac{\xi_k}{k!} (s-1)^{k-1} \quad (|s-1| < 1),$$

On the other hand, the function  $\Phi$  defined by (3) is an entire function of  $s$  with  $\Phi(1) = L_0 = 0$ , and the definition of  $L_k$  as  $\Phi^{(k)}(1)$  implies that the Taylor expansion of  $\Phi(s)$  at  $s = 1$  is given by

$$\Phi(s) = \sum_{k=1}^{\infty} \frac{L_k}{k!} (s-1)^k.$$

The Laurent expansion of  $\Gamma(1-s)\Phi(s)$  then follows by Cauchy product:

$$\begin{aligned} \Gamma(1-s)\Phi(s) &= -L_1 - \left(\frac{1}{2}L_2 + \gamma L_1\right)(s-1) \\ &\quad + \frac{1}{2}\left(-\frac{1}{3}L_3 - \gamma L_2 - \xi_2 L_1\right)(s-1)^2 \\ &\quad - \frac{1}{6}\left(\frac{1}{4}L_4 + \gamma L_3 + \frac{3}{2}\xi_2 L_2 + (2\zeta(3) + 3\gamma\zeta(2) + \gamma^3)L_1\right)(s-1)^3 + \dots \end{aligned} \quad (9)$$

c) The Taylor expansion of  $\zeta(s+1)$  at  $s=1$  can be written as follows:

$$\zeta(s+1) = \zeta(2) + \sum_{k=1}^{\infty} (-1)^k \frac{\omega_k}{k!} (s-1)^k. \quad (10)$$

By assembling equations (8)–(10) above, we first obtain (by identifying the constant term) the relation  $\gamma_{H,0} = -\gamma_1 - \zeta(2) - L_1$  which, thanks to (1), gives (5). In the same way, the general formulae (6) and (7) are easily derived by identifying the coefficients of higher degree.  $\square$

*Remark 1.* Using the notations of [15], let  $\gamma_k^{[2]}$  be the  $k$ th Stieltjes constant of the height one double zeta function  $\zeta(s,1) = \sum_{n>m>0} \frac{1}{n^s m}$ . In view of the relation  $\zeta(s,1) + \zeta(s+1) = \zeta_H(s)$ , we deduce the simple relation [15, Equation (14)]:

$$\gamma_{H,k} = \gamma_k^{[2]} + \omega_k \quad (k \geq 0).$$

Hence formulae (6) and (7) also apply to these constants with only a slight modification. Moreover, from a theorem of Ramanujan [7, Theorem 4], we have the identity

$$\omega_k = k! + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_{k+n} \quad (k \geq 1).$$

*Remark 2.* The constants  $\xi_n$  involved in formulae (6)–(7) have a polynomial expression in terms of Euler's constant  $\gamma$  and  $\zeta(k)$  for  $2 \leq k \leq n$  [2, 6]. More precisely,

$$\xi_n = n! P_n(\gamma, \zeta(2), \dots, \zeta(n)) \quad (n \geq 2),$$

where, for arbitrary  $n$ , the polynomials  $P_n$  are defined by the generating function

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, \dots, x_n) t^n = \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right).$$

In particular, we have

$$\xi_2 = \gamma^2 + \zeta(2), \quad \xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3), \quad \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3(\zeta(2))^2 + 6\zeta(4).$$

### 3.2 New integral representations of $\zeta(n)$ and $\zeta'(n)$

Applied in a neighborhood of  $s = n$  (for an integer  $n \geq 2$ ), formula (2) provides a relation between  $\Phi(s)$  and  $\zeta_H(s)$  which allows us to derive nice evaluations of  $\Phi(n)$  and  $\Phi'(n)$ .

**Proposition 2.** For any integer  $n \geq 2$ , we have

$$\Phi(n) = (-1)^{n-1}(n-1)!\zeta(n), \quad (11)$$

and

$$\begin{aligned} \Phi'(n) &= (-1)^{n-1}(n-1)! \\ &\times \left\{ \zeta'(n) + \zeta(n)\psi(n) - \frac{1}{2}n\zeta(n+1) + \frac{1}{2}\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1) \right\}. \end{aligned} \quad (12)$$

*Proof of Proposition 2.* For  $n \geq 2$ , we can write

$$\begin{aligned} \pi \cot(\pi s)\zeta(s) &= \frac{\zeta(n)}{s-n} + \zeta'(n) + O(s-n), \\ \Gamma(1-s) &= \frac{(-1)^n}{(n-1)!} \left( \frac{1}{s-n} + (\gamma - H_{n-1}) \right) + O(s-n), \end{aligned}$$

and

$$\Gamma(1-s)\Phi(s) = \frac{(-1)^n}{(n-1)!} \frac{\Phi(n)}{s-n} + \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + O(s-n).$$

By applying formula (2) around  $s = n$ , we obtain the equation

$$\begin{aligned} \zeta_H(n) + O(s-n) &= \left( \zeta(n) + \frac{(-1)^n}{(n-1)!}\Phi(n) \right) \frac{1}{s-n} \\ &+ \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + \zeta'(n) + \zeta(n+1) + O(s-n), \end{aligned}$$

which allows us to derive the identities  $\zeta(n) = \frac{(-1)^{n+1}}{(n-1)!}\Phi(n)$ , and

$$\zeta_H(n) = \frac{(-1)^n}{(n-1)!} (\Phi'(n) - \Phi(n)\psi(n)) + \zeta'(n) + \zeta(n+1).$$

Formulae (11) and (12) then follow using Euler's formula for  $\zeta_H(n)$ .  $\square$

We deduce from Proposition 2, the following corollary:



**Corollary 1.** For any integer  $n \geq 2$ , we have

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left(\text{Log}(1 + e^{ix})\right)^{n-1} dx. \quad (13)$$

and

$$\begin{aligned} \zeta'(n) &= \frac{1}{2} n \zeta(n+1) - \zeta(n) \psi(n) - \frac{1}{2} \sum_{k=1}^{n-2} \zeta(n-k) \zeta(k+1) \\ &\quad + \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log} \left(\text{Log}(1 + e^{ix})\right) \left(\text{Log}(1 + e^{ix})\right)^{n-1} dx. \end{aligned} \quad (14)$$

*Remark 3.* It should be noted that our formula (13) for  $\zeta(n)$  is similar to but simpler than the one given by [3, Equation (2.6)] which, after the substitution  $z = \text{Log}(1 + e^{ix})$ , translates into

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Log} \left(\text{Log}(1 + e^{ix})\right) \left(\text{Log}(1 + e^{ix})\right)^{n-1} dx.$$

**Example 2.** Applying the previous formulas to the simplest cases  $n = 2$  and  $n = 3$ , we get

$$\begin{aligned} \zeta(2) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log}(1 + e^{ix}) dx, \\ \zeta(3) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} ix \text{Log}^2(1 + e^{ix}) dx, \\ \zeta'(2) &= \zeta(3) - \zeta(2)(1 - \gamma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log} \left(\text{Log}(1 + e^{ix})\right) \text{Log}(1 + e^{ix}) dx. \end{aligned}$$

### 3.3 Link with the Glasser-Manna function

**Definition 1.** The Glasser-Manna function  $M$  [9, 11] is defined by

$$M(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}(e^{-z}(1 + e^{ix}))} dx = \frac{4}{\pi} \int_0^{\pi/2} \frac{y^2}{y^2 + \ln^2(2e^{-z} \cos(y))} dy.$$

It is analytic for  $z$  in the disc  $D(0, \ln 2)$  and verifies the relation

$$M^{(k)}(0) = k! \Phi(-k) \quad (k \geq 0). \quad (15)$$

*Remark 4.* Let  $\mathcal{L}\psi(x+1)$  be the Laplace transform of the function  $x \mapsto \psi(x+1)$ , the following relation can be shown:

$$M(z) = \frac{\gamma}{z} + \mathcal{L}\psi(x+1)(z) - \frac{1}{e^{-z} - 1} \left(z + \text{Log}(1 - e^{-z})\right) \quad (0 < |z| < \ln(2)).$$

Applied in a neighborhood of  $s = -n$ , formula (2) provides a relation between  $\Phi(s)$  and  $\zeta_H(s)$  which allows us to give a nice evaluation of  $\Phi(-n)$  for any non-negative integer  $n$ .

**Proposition 3.** We have

$$2\Phi(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1 + e^{ix})} dx = \ln(2\pi) - \gamma + 1, \quad (16)$$

and

$$\begin{aligned} 2\Phi'(0) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix \operatorname{Log}(\operatorname{Log}(1 + e^{ix}))}{\operatorname{Log}(1 + e^{ix})} dx \\ &= \gamma_1 + \frac{1}{2} \ln^2(2\pi) + \frac{1}{2} \gamma^2 - \gamma - \gamma \ln(2\pi) - \frac{7}{4} \zeta(2) + 2\beta, \end{aligned} \quad (17)$$

where  $\beta$  is the linear coefficient in the Laurent expansion of  $\zeta_H$  at  $s = 0$ .

*Proof of Proposition 3.* Around 0, we have the decomposition given by (2):

$$\zeta_H(s) = \zeta(s + 1) + \pi \cot(\pi s) \zeta(s) + \Gamma(1 - s) \Phi(s),$$

and the expansion [5, Equation (8)]

$$\zeta_H(s) = \frac{1}{2s} + \frac{1}{2} \gamma + \frac{1}{2} + \beta s + O(s^2).$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s + 1) &= \frac{1}{s} + \gamma - \gamma_1 s + O(s^2), \\ \pi \cot(\pi s) \zeta(s) &= -\frac{1}{2s} - \frac{1}{2} \ln(2\pi) + \left(\frac{1}{2} \gamma_1 - \frac{1}{4} \ln^2(2\pi) + \frac{1}{4} \gamma^2 + \frac{7}{8} \zeta(2)\right) s + O(s^2), \\ \Gamma(1 - s) \Phi(s) &= \Phi(0) + (\Phi'(0) + \gamma \Phi(0)) + O(s^2). \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand side of (2), we deduce the equation

$$\frac{1}{2} \gamma + \frac{1}{2} = \gamma - \frac{1}{2} \ln(2\pi) + \Phi(0),$$

which is equivalent to (16). In the same way, formula (17) is derived by identifying the linear coefficient in the expansion of the right-hand side of (2).  $\square$

*Remark 5.* The constant  $\beta$  which occurs in formula (17) can be evaluated using [15, Corollary 4.2]; more precisely, we have

$$\beta = 1 + \frac{1}{2}\gamma - \frac{1}{4}\gamma^2 - \gamma_1 + \frac{1}{4}\zeta(2) - \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^2} = 1.589935\dots,$$

where the  $b_n$  are the Bernoulli numbers of the second kind defined by way of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n x^n \quad (|x| < 1).$$

**Proposition 4.** We have

$$\Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^2(1+e^{ix})} dx = \ln(\mathcal{A}) - \frac{1}{12}\gamma + \frac{7}{24}, \quad (18)$$

where  $\ln(\mathcal{A}) = \frac{1}{12} - \zeta'(-1)$  and, more generally, for all integers  $n \geq 2$ , we have

$$(2n-1)! \Phi(1-2n) = \ln(\mathcal{A}_n) - \frac{B_{2n}}{2n}\gamma + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2}, \quad (19)$$

with  $\ln(\mathcal{A}_n) := \frac{H_{2n-1} B_{2n}}{2n} - \zeta'(1-2n)$ . The numbers  $\mathcal{A}_n$  for  $n \geq 2$  are called *generalized Glaisher-Kinkelin constants* (see [13] for more details).

**Example 3.** In particular,

$$\Phi(-3) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^4(1+e^{ix})} dx = \frac{1}{6} \ln(\mathcal{A}_2) + \frac{1}{720}\gamma + \frac{1}{320}.$$

*Proof of Proposition 4.* Around  $s = -1$ , we have the expansion [5, Equation (9)]

$$\zeta_H(s) = -\frac{1}{12(s+1)} - \frac{1}{12}\gamma - \frac{1}{8} + \mathcal{O}(s+1).$$

On the other side, we have

$$\begin{aligned} \zeta(s+1) &= -\frac{1}{2} + \mathcal{O}(s+1), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-1) + \mathcal{O}(s+1), \\ \Gamma(1-s)\Phi(s) &= \Phi(-1) + \mathcal{O}(s+1). \end{aligned}$$

Formula (18) is then deduced by identifying the constant coefficient in the expansion of the right-hand member of (2). In the same way, from [5, Proposition 2], the Laurent expansion around  $s = 1 - 2n$  for  $n \geq 2$  is given by

$$\zeta_H(s) = \frac{\zeta(1-2n)}{s+2n-1} - \frac{B_{2n}}{2n}\gamma + \frac{H_{2n-1}B_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} + O(s+2n-1)$$

which, by the same method, allows us to deduce formula (19).  $\square$

**Proposition 5.** For all positive integers  $n$ , we have

$$(2n)! \Phi(-2n) = -\zeta'(-2n) + (2n+1) \frac{B_{2n}}{4n}, \quad (20)$$

from which follows the reflection formula:

$$2(2n)! \Phi(-2n) = (-1)^{n+1} (2\pi)^{-2n} \Phi(2n+1) + (2n+1) \frac{B_{2n}}{2n}. \quad (21)$$

**Example 4.** In particular,

$$2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^3(1+e^{ix})} dx = -\zeta'(-2n) + \frac{1}{8} = \frac{1}{4\pi^2} \zeta(3) + \frac{1}{8}.$$

*Proof of Proposition 5.* Around  $s = -2n$ , we have by Matsuoka's formula

$$\zeta_H(s) = (2n-1) \frac{B_{2n}}{4n} + O(s+2n).$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s+1) &= -\frac{B_{2n}}{2n} + O(s+2n), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-2n) + O(s+2n), \\ \Gamma(1-s)\Phi(s) &= (2n)! \Phi(-2n) + O(s+2n). \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (2), we obtain formula (20). Moreover, the well-known identity

$$-2\zeta'(-2n) = (-1)^{n+1} (2\pi)^{-2n} (2n)! \zeta(2n+1)$$

and the relation  $(2n)! \zeta(2n+1) = \Phi(2n+1)$  given by (11) enables to deduce formula (21) from (20).  $\square$

## 4 Appendix: Evaluation of Glasser-Manna integrals by means of shifted Mascheroni series

In this appendix, we reinterpret some of the previous formulas in terms of shifted Mascheroni series. For any positive integer  $k$ , we consider the series

$$\sigma_k := \sum_{n=k+1}^{\infty} \frac{|b_n|}{n-k},$$

where  $b_n$  are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \dots$$

These series  $\sigma_k$  are called *shifted Mascheroni series*, they have been studied in detail in [8]. The nice identity

$$\sigma_1 = \frac{1}{2} \ln(2\pi) - \frac{1}{2} \gamma - \frac{1}{2}$$

is well-known [8, Proposition 2] and allows us to deduce from formula (16) an evaluation of  $\Phi(0)$  in terms of  $\sigma_1$ :

$$M(0) = \Phi(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}(1+e^{ix})} dx = \sigma_1 + 1 = 1.13033\dots \quad (\text{A1})$$

This evaluation was first given by Glasser and Manna [11, Proposition 3.3] in a slightly different but equivalent form. More generally, the following formula:

$$\sum_{k=1}^n (-1)^{n+k} k! S(n, k) \sigma_{k+1} = -\zeta'(-n) - \frac{B_{n+1}}{n+1} \left( \gamma + \frac{1}{n+1} \right) \quad (n \geq 1), \quad (\text{A2})$$

with  $S(n, k)$  denoting the Stirling numbers of the second kind [8, Proposition 3], allows us to give reinterpretations of formulas (18), (19) and (20) in terms of shifted Mascheroni series  $\sigma_k$ . In this way, we obtain

$$\begin{aligned} (2n-1)! \Phi(1-2n) &= M^{(2n-1)}(0) \\ &= \sum_{k=1}^{2n-1} (-1)^{k+1} k! S(2n-1, k) \sigma_{k+1} + \frac{B_{2n} H_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} \end{aligned} \quad (n \geq 1), \quad (\text{A3})$$

and

$$(2n)! \Phi(-2n) = M^{(2n)}(0) = \sum_{k=1}^{2n} (-1)^k k! S(2n, k) \sigma_{k+1} + (2n+1) \frac{B_{2n}}{4n} \quad (n \geq 1). \quad (\text{A4})$$

In particular, we have

$$M'(0) = \Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^2(1 + e^{ix})} dx = \sigma_2 + \frac{5}{12} = 0.49232 \dots$$

and

$$M^{(2)}(0) = 2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^3(1 + e^{ix})} dx = 2\sigma_3 - \sigma_2 + \frac{1}{8} = 0.15544 \dots$$

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