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# Miscellaneous series with Cauchy and harmonic numbers and their interpretation as Ramanujan summation

Marc-Antoine Coppo\*

*Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France*

**Abstract** We provide an overview of several series identities involving Cauchy numbers and harmonic numbers, all of which are closely linked to certain alternating series with zeta (or harmonic zeta) values; we then give, for each of them, their interpretation in terms of Ramanujan summation with the hope that this unusual interpretation of still little-known formulas should be useful for further research on the topic.

**Keywords** Cauchy numbers; harmonic numbers; binomial transform; series with zeta values; Ramanujan summation of series.

**Mathematics Subject Classification (2020)** 05A19, 11B75, 11M06, 40G99.

## 1 Introduction

A decade ago, we showed how Ramanujan's method of summation of series could be used to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and values of the Riemann zeta function at positive integers [4]. Recently, we have improved this method by introducing a natural generalization of the classical Roman harmonic numbers [8] and by systematically using a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series (see [6] for details). In some cases, an alternative efficient method for generating such identities, based on a similar binomial transformation formula, is described in [1]. By

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\*Corresponding author. *Email address:* coppo@unice.fr

combining these two methods and reversing the procedure, we are able to provide an interesting interpretation in terms of Ramanujan summation of almost all the identities presented here (see Section 5).

## 2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers (also known as Bernoulli numbers of the second kind) and introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers  $\{\lambda_n\}_{n \geq 1}$  are defined explicitly by the formula

$$\lambda_n := \int_0^1 x(1-x) \cdots (n-1-x) dx.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!(n-k)} = \frac{1}{n} \quad (n \geq 2).$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{1}{4}, \lambda_4 = \frac{19}{30}, \lambda_5 = \frac{9}{4}, \lambda_6 = \frac{863}{84}, \text{ etc.}$$

The non-alternating Cauchy numbers  $\lambda_n$  are closely linked to the Bernoulli numbers of the second kind  $b_n$  first introduced by Jordan [7] through the relation

$$\lambda_n = n! |b_n| \quad (n \geq 1).$$

Otherwise, with the current notation  $c_n$  used in [1], we have the simple relation

$$\lambda_n = (-1)^{n-1} c_n \quad (n \geq 1).$$

We also recall the convergence to 1 of the series  $\sum_{n \geq 1} \frac{\lambda_n}{n!}$ .

b) The classical harmonic numbers  $\{H_n\}_{n \geq 1}$  are defined by

$$H_n = \sum_{j=1}^n \frac{1}{j} = \psi(n+1) + \gamma,$$

where  $\psi$  denotes the digamma function and  $\gamma = -\psi(1)$  is the Euler constant.

c) For any integer  $k \geq 1$ , the generalized harmonic numbers  $\{H_n^{(k)}\}_{n \geq 1}$  are defined by  $H_n^{(1)} = H_n$ , and

$$H_n^{(k)} = \sum_{j=1}^n \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!} \partial^{k-1} \psi(n+1) + \zeta(k) \quad (k \geq 2).$$

d) For any integer  $k \geq 0$ , the Roman harmonic numbers  $\{H_{n,k}\}_{n \geq 1}$  are defined by  $H_{n,0} = 1$ , and

$$H_{n,k} = \sum_{n \geq j_1 \geq \dots \geq j_k \geq 1} \frac{1}{j_1 j_2 \cdots j_k} \quad (k \geq 1).$$

The Roman harmonic numbers<sup>1</sup> can be expressed as polynomials in the generalized harmonic numbers  $H_n, H_n^{(2)}, \dots, H_n^{(k)}$  (see [4, Eq. (18)], [8, Eq. (29)]). More precisely, we have  $H_{n,1} = H_n$ , and

$$H_{n,k} = \frac{1}{k!} (H_n)^k + \dots + \frac{1}{k} H_n^{(k)} = P_k(H_n, \dots, H_n^{(k)}) \quad (k \geq 2),$$

where  $P_k$  are the modified Bell polynomials [4, Def. 2]. In particular,

$$H_{n,2} = \sum_{k=1}^n \frac{H_k}{k} = \frac{1}{2} (H_n)^2 + \frac{1}{2} H_n^{(2)} = P_2(H_n, H_n^{(2)}).$$

A natural generalization of the ordinary Roman harmonic numbers noted  $H_{n,k}^{(r)}$  such that  $H_{n,1}^{(r)} = H_n^{(r)}$  was also introduced in [6] (see [6, Def. 2]). It is given by the following expression:

$$H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \dots \geq j_k \geq 1} \frac{1}{j_1 j_2 \cdots j_k^r} \quad (k \geq 1, r \geq 1).$$

### 3 Overview of some known formulas

In this section, we mention a number of more or less known identities and make some comments about them.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) = \gamma \quad (1)$$

is a classical representation of  $\gamma$  due to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)! n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \{\zeta(n) - 1\} = \gamma + \log 2 - 1. \quad (2)$$

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1. Introduced three decades ago by S. Roman, G-C. Rota and D. Loeb (see [8] for historical details).

b) A non-trivial generalization of (1) is the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1), \quad (3)$$

where  $\gamma_1$  denotes the first Stieltjes constant. This intriguing identity is already known and appears in [3, 5, 6].

c) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \zeta(2) - 1 \quad (4)$$

is a fairly known representation of  $\zeta(2) = \frac{\pi^2}{6}$  which is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_{n,k}}{n! n} = \zeta(k+1) - \frac{1}{k} \quad (k \geq 1),$$

sometimes called *Hermite's formula* [4].

d) A non-trivial generalization of (4) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n! n} = \zeta(3) + \{\gamma + \log(2\pi) - 12 \log(A)\} \zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+2) \quad (5)$$

with  $A$  denoting the Glaisher-Kinkelin constant

$$A = \lim_{n \rightarrow \infty} \left\{ n^{-\frac{n^2}{2} - \frac{n}{2} - \frac{1}{12}} e^{\frac{n^2}{4}} \prod_{k=1}^n k^k \right\}.$$

This identity results directly from [6, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \log(2\pi) - 12 \log(A)) \zeta(2).$$

Furthermore, formula (5) admits a kind of reciprocal which is given by [6, Eq. (16)] in the case  $p = 2$ . More precisely, we have the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2} = \zeta(2) (12 \log(A) - \log(2\pi)) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+2). \quad (6)$$

*Remark 1.* When  $k$  is greater than 2, no explicit formula, even conjectural, appears to be known for the sum  $\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n! n}$ , nor for the reciprocal sum  $\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^k}$  (see however Remark 3 below for an interpretation of these sums in terms of Ramanujan summation).

## 4 Additional formulas

The following binomial formula is nothing else than a variant of [1, Prop. 1] which is an elementary but efficient tool for obtaining several series identities with Cauchy numbers. For appropriate analytic functions  $f$  with moderate growth, we have the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x) dx. \quad (7)$$

For instance, formulas (1) and (4) above can be easily derived from formula (7) using adequate functions  $f$  (see [1, Ex. 4]). We now give another new interesting identities that can also be deduced by this method.

e) Applying (7) with  $f(x) = \frac{\psi(x+1) + \gamma}{x+1}$ , and using the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1},$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \int_0^1 f(x) dx = \frac{1}{2} \zeta(2) + \log 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^n (\zeta(k) - 1) \right\} \quad (8)$$

which is a refinement of a formula previously given by Boyadzhiev [1, Ex. 5]. Moreover, subtracting (8) from (4) leads to the following new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)! n} = \frac{1}{2} \zeta(2) - \log 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^n (\zeta(k) - 1) \right\}. \quad (9)$$

This last formula is a modification of (4) quite similar to (2).

f) Applying (7) with  $f(x) = \frac{\psi(x+1) + \gamma}{x}$ , and using the binomial identity

$$H_n^{(2)} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{H_k}{k},$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \int_0^1 f(x) dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1). \quad (10)$$

Moreover, writing  $H_n^{(2)} = H_{n-1}^{(2)} + \frac{1}{n^2}$  and using (3) leads to another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \frac{1}{2} \zeta(2) - \frac{1}{2} \gamma^2 - \gamma_1. \quad (11)$$

If  $\zeta_H$  denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \quad (\operatorname{Re}(s) > 1),$$

then, thanks to [5, Eq. (14)], this last identity may also be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n). \quad (11 \text{ bis})$$

*Remark 2.* Very recently, Young [9] has shown the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1, \{1\}^{k-2}) \quad (k \geq 2)$$

which considerably generalizes (10). It should be noted that, in contrast to the series  $\sum_{n \geq 1} \frac{\lambda_n H_n^{(k)}}{n!}$ , the series  $\sum_{n \geq 1} \frac{\lambda_n H_n}{n!}$  is divergent since  $\frac{\lambda_n H_n}{n!} \sim \frac{1}{n \log(n)}$ .

## 5 Interpretation as Ramanujan summation

If  $\sum_{n \geq 1}^{\mathcal{R}}$  denotes the  $\mathcal{R}$ -sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial formula [6, Eq. (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} f(k) = \sum_{n \geq 1}^{\mathcal{R}} \frac{f(n)}{n}. \quad (12)$$

This formula allows us to give an interesting interpretation in terms of Ramanujan summation of almost all the series identities mentioned above.

a) From the identities

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} = 1 \quad \text{and} \quad \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{k}{k+1} = \frac{1}{n+1},$$

we derive respectively

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n}, \quad (\text{A})$$

and the shifted formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n+1}. \quad (\text{B})$$

b) From the reciprocal binomial identities

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \quad \text{and} \quad \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n,$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n}. \quad (\text{C})$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n^2}, \quad (\text{D})$$

c) From the reciprocal binomial identities

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} H_k^{(2)} = \frac{H_n}{n} \quad \text{and} \quad \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} = H_n^{(2)},$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}. \quad (\text{E})$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^2}, \quad (\text{F})$$

*Remark 3.* a) Formulas (A) and (C) are two particular cases of the more general relation [6, Eq. (12)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^k} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \quad (k \geq 1).$$



b) Formulas (D) and (F) are two particular cases of the more general relation [6, Eq. (11)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \quad (k \geq 1).$$

c) Formula (E) can be generalized by means of the generalized Roman harmonic numbers  $H_{n,k}^{(r)}$  introduced in [6]. Indeed, by [6, Eq. (12)], we have

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^k} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1}^{(2)}}{n} \quad (k \geq 2).$$

d) From the the self-reciprocal binomial identity

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1} = \frac{H_n}{n+1},$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)}. \quad (\text{G})$$

e) Let us now apply the formula (12) to the function

$$f(x) = \frac{\psi(x+1) + \gamma - 1}{x(x-1)},$$

which verifies

$$f(1) = \lim_{x \rightarrow 1} \frac{\psi(x+1) + \gamma - 1}{x(x-1)} = \zeta(2) - 1,$$

and

$$f(n) = \frac{H_n - 1}{n(n-1)} \quad (n \geq 2).$$

Then, by means of the binomial identity

$$\sum_{k=2}^n (-1)^{k-1} \binom{n}{k} \frac{1 - H_k}{k-1} = nH_n^{(2)} - n$$

which is the reciprocal of the identity [2, Eq. (5.24)]:

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} k H_k^{(2)} = \frac{1 - H_n}{n-1} \quad (n \geq 2),$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)}. \quad (\text{H})$$

Furthermore, subtracting (C) from (H), leads to the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n-1}}{n-1}. \quad (\text{I})$$

*Remark 4.* Very recently, Young [9] has established the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k+1)}}{n!} = \zeta(k+1) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k} - H_{n,k-1}}{n(n-1)} \quad (k \geq 1)$$

which is a deep generalization of our formula (H).

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