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On some constants related to the harmonic zeta function

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Abstract Using Ramanujan's summation method applied to a certain type of divergent series involving harmonic numbers and powers of logarithms, we define a variety of remarkable constants and study their properties. In particular, we clarify their close connection with the Stieltjes constants and the harmonic zeta function.

Keywords Harmonic numbers; Stieltjes constants; Zeta values; Bernoulli numbers; Gamma function; Ramanujan summation of divergent series.

1 Introduction

The harmonic zeta function ζ_H is defined for $\operatorname{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s},$$

where, for all integers $n \geq 1$,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Four decades ago, Apostol and Vu [1] and Matsuoka [12], completing Euler's original research [11], showed that this function could be continued as a meromorphic function with a double pole at $s = 1$, and simple poles at $s = 0$ and at odd negative integers. The special values of ζ_H at even negative integers are given by Matsuoka's formula:

$$2\zeta_H(-2p) = (1 - 2p)\zeta(1 - 2p) = (2p - 1)\frac{B_{2p}}{2p} \quad (p \geq 1),$$

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where the B_{2p} are the classical Bernoulli numbers. The Laurent expansion of the harmonic zeta function ζ_H around its double pole can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_{H,k} (s-1)^k \quad (0 < |s-1| < 1),$$

where $\gamma = -\Gamma'(1) = 0.577215\dots$ is Euler's constant, and the coefficients $\gamma_{H,k}$ are called *harmonic Stieltjes constants* by analogy with the classical Stieltjes constants. A common definition of the Stieltjes constants γ_k for arbitrary k is

$$\gamma_k = \lim_{s \rightarrow 1} \left\{ (-1)^k \zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}} \right\} \quad (k \geq 0),$$

where $\zeta^{(k)}(s)$ is the k th derivative of the Riemann zeta function [7]. In particular, γ_0 is Euler's constant γ . An explicit expression of $\gamma_{H,0}$ is given by the following formula [6, Equation (6)]:

$$\gamma_{H,0} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\Gamma^{(2)}(1) = 0.989055\dots \quad (1)$$

This nice expression arises from a special case of a general formula which applies to height one multiple zeta functions of arbitrary depth [13, Equation (28)].

Regarding the classical Stieltjes constants, we recall the well-known asymptotic representation of γ_k (see e.g. [7]):

$$\gamma_k = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{(\ln n)^k}{n} - \frac{(\ln N)^{k+1}}{k+1} \right\} \quad (k \geq 0).$$

From the point of view adopted here, the constant γ_k is interpreted as the \mathcal{R} -sum of the divergent series $\sum_{n \geq 1} \frac{(\ln n)^k}{n}$, i.e. the sum of the series in the sense of Ramanujan's summation method, following the exposition in [4] and [5, Chapter IX]. Thus, we have [4, p. 67], [5, p. 296]:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{(\ln n)^k}{n} = \gamma_k.$$

The existence of a similar asymptotic representation for $\gamma_{H,k}$:

$$\gamma_{H,k} = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{H_n (\ln n)^k}{n} - \frac{(\ln N)^{k+2}}{k+2} - \gamma \frac{(\ln N)^{k+1}}{k+1} \right\} \quad (k \geq 0),$$

which results from a generalization of a formula of Briggs and Buschman [2, 3], suggests that the harmonic Stieltjes constant $\gamma_{H,k}$ and the \mathcal{R} -sum $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^k}{n}$

are closely linked. As we will see, such a connection exists although it is far less simple than in the previous case (see Proposition 1 below). In earlier studies [6, Equation (4)], [8, Equation (5.1)], we have already encountered and drawn attention to the interesting relation:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1 = \gamma_{H,0} + \gamma_1 + \tau_1 - \zeta(2) = 0.529052\dots, \quad (2)$$

where

$$\tau_1 := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1) = 1.257746\dots$$

it should be noted that the constant τ_1 arises in several contexts in number theory (see e.g. [14] for a summary of these occurrences). The main purpose of this article is to determine to what extent this formula can be generalized. We start with the following definition:

Definition 1. For any positive integer k , the constant τ_k is defined by

$$\tau_k := \sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{n^k} \zeta(n+1).$$

Remark 1. Although not used afterwards, we also mention another expression of the constant τ_k which can be easily deduced from the definition above:

$$\tau_k = (-1)^k \sum_{n=1}^{\infty} \frac{1}{n} \text{Li}_k\left(-\frac{1}{n}\right) \quad (k \geq 1),$$

where Li_k denotes the k th polylogarithm. In particular, for $k = 1$, the formula

$$\tau_1 = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right)$$

is regained.

2 How the constants τ_k , γ_k and $\gamma_{H,k}$ are linked together

In this section, we clarify the link between the constants defined above. The forthcoming Proposition 1 successfully generalizes formula (2).

Proposition 1. For any positive integer k , we have the following relations:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^{2k-1}}{n} = \gamma_{H,2k-1} + \frac{\gamma_{2k}}{2k} + (2k-1)! \tau_{2k} + \sum_{j=1}^k \frac{(2k-1)!}{(2k-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \gamma_{2k-2j} \zeta(2j), \quad (3)$$

and

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^{2k}}{n} = \gamma_{H,2k} + \frac{\gamma_{2k+1}}{2k+1} + (2k)! \tau_{2k+1} + \sum_{j=1}^k \frac{(2k)!}{(2k+1-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \gamma_{2k+1-2j} \zeta(2j) - (2k)! \frac{(2^{2k+1}-1)}{2^{2k}} \zeta(2k+2). \quad (4)$$

In particular, we have

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n \ln n}{n} &= \gamma_{H,1} + \frac{1}{2} \gamma_2 + \tau_2 + \gamma \zeta(2), \\ \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^2}{n} &= \gamma_{H,2} + \frac{1}{3} \gamma_3 + 2\tau_3 + 2\gamma_1 \zeta(2) - \frac{7}{2} \zeta(4), \\ \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^3}{n} &= \gamma_{H,3} + \frac{1}{4} \gamma_4 + 6\tau_4 + 3\gamma_2 \zeta(2) + \frac{21}{2} \gamma \zeta(4). \end{aligned}$$

Proof of Proposition 1. The key formula to derive the general relations (3) and (4) is the splitting of $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s}$ given by [6, Theorem 1] which is recalled below:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \frac{\pi}{\sin(\pi s)} \zeta(s) + \int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx + \zeta_H(s), \quad (5)$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. This formula applies to all complex numbers s such that $\text{Re}(s) < 2$ and $s \neq 1, 0$, and $1 - 2k$ for each positive integer k . Fortunately, the Laurent expansion of each component in (5) can be written explicitly.

a) The expansion of $\frac{\pi}{\sin(\pi s)} \zeta(s)$ at $s = 1$ can be obtained as follows: first, we

write the successive equations:

$$\begin{aligned}
\frac{-\pi}{\sin(\pi s)} &= \frac{\pi}{\sin(\pi(s-1))} = \frac{\exp(i\pi(s-1))}{s-1} \frac{2i\pi(s-1)}{\exp(2i\pi(s-1)) - 1} \\
&= \frac{1}{s-1} \left(\sum_{k \geq 0} i^k \pi^k \frac{1}{k!} (s-1)^k \right) \left(\sum_{k \geq 0} i^k (2\pi)^k \frac{B_k}{k!} (s-1)^k \right) \\
&= \frac{1}{s-1} + \sum_{k \geq 1} (-1)^k \left(\sum_{j=0}^{2k} \binom{2k}{j} 2^j B_j \right) \frac{\pi^{2k}}{(2k)!} (s-1)^{2k-1},
\end{aligned}$$

where the B_j are the Bernoulli numbers. Euler's identity:

$$\zeta(2k) = (-1)^{k+1} 2^{2k-1} B_{2k} \frac{\pi^{2k}}{(2k)!} \quad (k \geq 1),$$

then allows us to rewrite this expansion as follows:

$$\frac{-\pi}{\sin(\pi s)} = \frac{1}{s-1} - \sum_{k \geq 1} \frac{2^{1-2k}}{B_{2k}} \sum_{j=0}^{2k} \binom{2k}{j} 2^j B_j \zeta(2k) (s-1)^{2k-1}.$$

Moreover, the latter expression can be simplified thanks to the identity

$$\sum_{j=0}^k \binom{k}{j} 2^j B_j = 2^k B_k \left(\frac{1}{2}\right) = 2(1 - 2^{k-1}) B_k \quad (k \geq 2).$$

Hence, the expansion of $\frac{\pi}{\sin(\pi s)}$ at $s = 1$ is given by

$$\frac{\pi}{\sin(\pi s)} = -\frac{1}{s-1} - \sum_{k=1}^{\infty} \frac{2^{2k-1} - 1}{2^{2k-2}} \zeta(2k) (s-1)^{2k-1}.$$

On the other hand, the expansion of $\zeta(s)$ at $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k,$$

where γ_k are the Stieltjes constants. The expansion of $\frac{\pi}{\sin(\pi s)} \zeta(s)$ is then obtained by Cauchy product. We have

$$\begin{aligned}
\frac{\pi}{\sin(\pi s)} \zeta(s) &= -\frac{1}{(s-1)^2} - \frac{\gamma}{(s-1)} + \gamma_1 - \zeta(2) \\
&\quad - \left(\frac{1}{2} \gamma_2 + \gamma \zeta(2) \right) (s-1) + \left(\frac{1}{6} \gamma_3 + \gamma_1 \zeta(2) - \frac{7}{4} \zeta(4) \right) (s-1)^2 \\
&\quad - \left(\frac{1}{24} \gamma_4 + \frac{1}{2} \gamma_2 \zeta(2) + \frac{7}{4} \gamma \zeta(4) \right) (s-1)^3 \\
&\quad + \left(\frac{1}{120} \gamma_5 + \frac{1}{6} \gamma_3 \zeta(2) + \frac{7}{4} \gamma_1 \zeta(4) - \frac{31}{16} \zeta(6) \right) (s-1)^4 - \dots \quad (6)
\end{aligned}$$

In (6), the coefficient of $(s-1)^{2k}$ and $(s-1)^{2k-1}$ are respectively

$$\frac{\gamma_{2k+1}}{(2k+1)!} + \sum_{j=1}^k \frac{\gamma_{2k-2j+1}}{(2k-2j+1)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \zeta(2j) - \frac{(2^{2k+1}-1)}{2^{2k}} \zeta(2k+2),$$

and

$$-\frac{\gamma_{2k}}{(2k)!} - \sum_{j=1}^k \frac{\gamma_{2k-2j}}{(2k-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \zeta(2j).$$

b) The well-known Taylor series expansion of the digamma function:

$$\psi(x+1) + \gamma = \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(n+1) x^n \quad (|x| < 1),$$

allows us write

$$\int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n-(s-1)}.$$

This leads to the following expansion:

$$\begin{aligned} \int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx &= \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n^{k+1}} \right) (s-1)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times k! \tau_{k+1} (s-1)^k \quad (|s-1| < 1). \end{aligned} \quad (7)$$

c) To obtain the expansion of $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s}$ at $s=1$, we proceed as follows: first we write the identities

$$\frac{H_n}{n^s} = \frac{H_n}{n} e^{-(s-1)\ln n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{H_n (\ln n)^k}{n} (s-1)^k,$$

so that

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{n \geq 1}^{\mathcal{R}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{H_n (\ln n)^k}{n} (s-1)^k \right).$$

Moreover, [4, Theorem 9] allows us interchange $\sum_{n \geq 1}^{\mathcal{R}}$ and $\sum_{k=0}^{\infty}$. This leads to the expansion:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^k}{n} \right) (s-1)^k. \quad (8)$$

By combining the previous expansions (6), (7), and (8), we recover our formula (2) and we also obtain the desired formulas (3) and (4). \square

3 Explicit expression of $\gamma_{H,k}$

If b_n denotes the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \dots \quad (|x| < 1),$$

then the following formula is well-known [4, Equation 4.29], [10, Equation 16]:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_j = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.529052\dots$$

Hence formula (2) can be rewritten as follows:

$$\gamma_{H,0} = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} - \gamma_1 - \tau_1 + \zeta(2).$$

More generally, for each natural number k , we derive from [10, Equation (13)] the following identity:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\ln n)^k}{n} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_j (\ln j)^k \quad (k \geq 0). \quad (9)$$

Using Proposition 1, we then obtain an expression of $\gamma_{H,k}$ to be compared with a similar formula for γ_k which was given in [9], namely:

$$\gamma_k = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} (\ln j)^k \quad (k \geq 0).$$

Corollary 1. For k odd, we have

$$\begin{aligned} \gamma_{H,k} &= \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_j (\ln j)^k \\ &\quad - \frac{\gamma_{k+1}}{k+1} - k! \tau_{k+1} - \sum_{j=1}^{k+1/2} \frac{(2j-1)! (2^{2j-1} - 1)}{2^{2j-2}} \binom{k}{2j-1} \gamma_{k+1-2j} \zeta(2j), \end{aligned} \quad (10)$$

and for k even, we have

$$\begin{aligned} \gamma_{H,k} &= \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_j (\ln j)^k \\ &\quad - \frac{\gamma_{k+1}}{k+1} - k! \tau_{k+1} - \sum_{j=1}^{k/2} \frac{(2j-1)! (2^{2j-1} - 1)}{2^{2j-2}} \binom{k}{2j-1} \gamma_{k+1-2j} \zeta(2j) \\ &\quad \quad \quad + k! \frac{(2^{k+1} - 1)}{2^k} \zeta(k+2). \end{aligned} \quad (11)$$

Example 1.

$$\gamma_{H,1} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_j \ln j - \frac{\gamma_2}{2} - \tau_2 - \gamma\zeta(2) = 0.400761\dots$$

4 Another possible generalization

Finally, we mention another rather natural way of extending formula (2) by replacing the classical harmonic numbers H_n with the Roman harmonic numbers $H_{n,k}$ [14, Equation (2.17)]. More precisely, for an arbitrary $k \geq 1$, let

$$H_{n,k} := \sum_{n \geq n_1 \geq \dots \geq n_k \geq 1} \frac{1}{n_1 n_2 \dots n_k} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} j^{-k}.$$

If

$$\tau_1^{[k]} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1, \underbrace{1, \dots, 1}_{k-1}),$$

and if $\gamma_1^{[k]}$ denotes the first Stieltjes constant of the height one zeta function of depth k , $\zeta(s, \underbrace{1, \dots, 1}_{k-1})$, then the following relation can be established [13, Theorem

6.1]:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k}}{n} = \sum_{n=1}^{\infty} \frac{|b_n|}{n^{k+1}} = P_{k+1}(\gamma, -\zeta(2), \dots, (-1)^k \zeta(k+1)) + \gamma_1^{[k]} + \tau_1^{[k]},$$

where the P_n are the modified Bell polynomials defined by means of their generating function:

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x_1, \dots, x_n) t^n &= \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) \\ &= 1 + x_1 t + \frac{1}{2}(x_1^2 + x_2) t^2 + \frac{1}{6}(x_1^3 + 3x_1 x_2 + 2x_3) t^3 + \dots \end{aligned}$$

In the simplest case $k = 1$, we have $H_{n,1} = H_n$, $\tau_1^{[1]} = \tau_1$, $\gamma_1^{[1]} = \gamma_1$, and $P_2(\gamma, -\zeta(2)) = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)$, and so formula (2) is regained.

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