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# On some constants related to the harmonic zeta function

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**Abstract** We study a wide range of constants which are closely related to the harmonic zeta function  $\zeta_H$ . Furthermore, by means of a new representation of  $\zeta_H$ , we derive a new integral representation for  $\zeta(n)$ , as well as new evaluations for a variety of integrals of the same type as those previously considered by Glasser and Manna in connection with the Laplace transform of the digamma function.

**Keywords** Harmonic zeta function; Stieltjes constants; zeta values; digamma function; Glasser-Manna integrals; Ramanujan summation of series.

## 1 Introduction

Let  $\zeta_H$  be the harmonic zeta function defined for  $\operatorname{Re}(s) > 1$  by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s},$$

where, for all  $n \geq 1$ ,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Four decades ago, Apostol and Vu [2] showed that this function could be continued as a meromorphic function with a double pole at  $s = 1$ , and an infinite number of simple poles at  $s = 0$  and  $s = 1 - 2k$  for each integer  $k \geq 1$ . The Laurent expansion of the harmonic zeta function  $\zeta_H$  around its double pole  $s = 1$  can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_H^{(k)} (s-1)^k \quad (0 < |s-1| < 1),$$

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where  $\gamma = -\Gamma'(1)$  is Euler's constant, and the coefficients  $\gamma_H^{(k)}$  are called *harmonic Stieltjes constants* by analogy with the classical Stieltjes constants  $\gamma_k$ . A common definition of these constants is

$$\gamma_k = \lim_{s \rightarrow 1} \left\{ (-1)^k \zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}} \right\} \quad (k \geq 0),$$

where  $\zeta^{(k)}(s)$  is the  $k$ th derivative of the Riemann zeta function. From now on and throughout the text, we will use the lighter notation  $\gamma^{(k)}$  in place of  $\gamma_H^{(k)}$ . An explicit expression of  $\gamma^{(0)}$  is given by the following formula [10, Equation (6)]:

$$\gamma^{(0)} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\Gamma''(1). \quad (1)$$

It is worth noting that this nice formula can be generalized to the case of height 1 multiple zeta functions  $\zeta(s, 1, \dots, 1)$  [17, Corollary 4.1]. Regarding the classical Stieltjes constants, we recall the well-known asymptotic formula

$$\gamma_k = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{(\log n)^k}{n} - \frac{(\log N)^{k+1}}{k+1} \right\} \quad (k \geq 0).$$

From our point of view, it should be noted that  $\gamma_k$  is nothing else than the  $\mathcal{R}$ -sum of the divergent series  $\sum_{n \geq 1} \frac{(\log n)^k}{n}$  [9, p. 67], where the  $\mathcal{R}$ -sum of a series denotes the sum of the series in the sense of Ramanujan's summation method, following the exposition in [9]. The existence of a similar asymptotic formula for  $\gamma^{(k)}$ :

$$\gamma^{(k)} = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{H_n (\log n)^k}{n} - \frac{(\log N)^{k+2}}{k+2} - \gamma \frac{(\log N)^{k+1}}{k+1} \right\} \quad (k \geq 0),$$

which results from an extension of a formula of Briggs and Buschman [4, 8], strongly suggests that the constant  $\gamma^{(k)}$  is closely linked to the  $\mathcal{R}$ -sum of the divergent series  $\sum_{n \geq 1} \frac{H_n (\log n)^k}{n}$ . As we shall see, this is indeed the case: such a relation exists although it is more intricate than in the classical case (see section 2). In earlier studies [9, Equation (3.23)], [10, Equation (4)], we have already drawn attention to the intriguing relation:

$$r_0 = \gamma_1 + \tau_1 + \gamma^{(0)} - \zeta(2) = \gamma_1 + \tau_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2), \quad (2)$$

where

$$r_0 := \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n} \quad \text{and} \quad \tau_1 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n} = - \sum_{n=2}^{\infty} \zeta'(n).$$

In the next section, we will see to what extent this formula (2) can be generalized (see Proposition 1). We also mention that another natural way of extending formula (2), involving Roman harmonic numbers  $H_{n,k}$  instead of the classic harmonic numbers, was recently explored by Young [17, Theorem 6.1 (d)].

In the subsequent section, we will give a new representation of  $\zeta_H(s)$  which is a subtle improvement of an earlier formula due to Boyadzhiev, Gadiyar and Padma [7] (see Theorem 1), and use it to derive new expressions of the harmonic Stieltjes constants  $\gamma^{(k)}$  (see Corollary 1). Moreover, we also derive from this representation of  $\zeta_H$  a new integral formula for  $\zeta(n)$  (see Corollary 2) as well as new evaluations for a variety of integrals similar to those introduced earlier by Glasser and Manna [1, 15] in connection with the Laplace transform of the digamma function (see Corollaries 4–6).

## 2 Connection between $\gamma^{(k)}$ and $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\log n)^k}{n}$

In this section, we explain how the constants  $\gamma^{(k)}$  and  $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\log n)^k}{n}$  are linked together. The forthcoming Proposition 1 successfully generalizes the previous relation (2).

**Proposition 1.** Let  $k$  be a natural number. If  $r_k$  and  $\tau_k$  are respectively defined by

$$r_k := \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\log n)^k}{n} \quad (k \geq 0),$$

and

$$\tau_k := \sum_{n=1}^{\infty} (-1)^{n+k} \frac{\zeta(n+1)}{n^k} \quad (k \geq 1),$$

then

$$r_0 = \gamma_1 + \tau_1 + \gamma^{(0)} - \zeta(2),$$

and the following relations hold true for all positive values of  $k$ :

$$r_{2k-1} = \frac{\gamma_{2k}}{2k} + (2k-1)! \tau_{2k} + \gamma^{(2k-1)} + \sum_{n=1}^k \frac{(2k-1)!}{(2k-2n)!} \times \frac{(2^{2n-1} - 1)}{2^{2n-2}} \gamma_{2k-2n} \zeta(2n), \quad (3)$$

and

$$r_{2k} = \frac{\gamma_{2k+1}}{2k+1} + (2k)! \tau_{2k+1} + \gamma^{(2k)} - (2k)! \frac{(2^{2k+1} - 1)}{2^{2k}} \zeta(2k+2) + \sum_{n=1}^k \frac{(2k)!}{(2k-2n+1)!} \times \frac{(2^{2n-1} - 1)}{2^{2n-2}} \gamma_{2k-2n+1} \zeta(2n). \quad (4)$$

In particular,

$$\begin{aligned} r_1 &= \frac{1}{2}\gamma_2 + \tau_2 + \gamma^{(1)} + \gamma\zeta(2), \\ r_2 &= \frac{1}{3}\gamma_3 + 2\tau_3 + \gamma^{(2)} - \frac{7}{2}\zeta(4) + 2\gamma_1\zeta(2). \end{aligned}$$

**Example 1.** Numerical evaluations of  $r_k$ ,  $\tau_k$  and  $\gamma^{(k)}$  for small values of  $k$  are given below. Note that the constant  $\tau_0$  is undefined.

$$\begin{array}{lll} r_0 = 0.529052\dots & \tau_1 = 1.257746\dots & \gamma^{(0)} = 0.989055\dots \\ r_1 = -0.078850\dots & \tau_2 = -1.424248\dots & \gamma^{(1)} = 0.400761\dots \\ r_2 = -0.008095\dots & \tau_3 = 1.523800\dots & \gamma^{(2)} = 0.971304\dots \end{array}$$

**Remark 1.** Although not used afterwards, we also mention another expression of the constant  $\tau_k$  which can be easily deduced from its definition:

$$\tau_k = (-1)^k \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Li}_k\left(-\frac{1}{n}\right) \quad (k \geq 1),$$

where  $\operatorname{Li}_k$  denotes the  $k$ th polylogarithm. In particular, for  $k = 1$ , the formula

$$\tau_1 = \sum_{n=1}^{\infty} \frac{1}{n} \log\left(1 + \frac{1}{n}\right)$$

is regained. Note that the definition of  $\tau_k$  given here differs from that given in [10] when  $k > 1$ .

*Proof of Proposition 1.* The key formula to derive the general relations (3) and (4) is the decomposition of  $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s}$  given by [10, Theorem 1] which is reproduced below:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \frac{\pi}{\sin(\pi s)} \zeta(s) + \int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx + \zeta_H(s), \quad (5)$$

where  $\psi(x) = \frac{d}{dx} \log \Gamma(x)$  is the digamma function. This formula applies to all complex numbers  $s$  such that  $\operatorname{Re}(s) < 2$  and  $s \neq 1, 0, 1 - 2k$  for each integer  $k > 0$ . Fortunately, the Laurent expansion of each component in (5) can be written explicitly.

a) The expansion of  $\frac{\pi}{\sin(\pi s)} \zeta(s)$  at  $s = 1$  can be obtained as follows: first, we write

the successive equations:

$$\begin{aligned}
\frac{-\pi}{\sin(\pi s)} &= \frac{\pi}{\sin(\pi(s-1))} = \frac{\exp(i\pi(s-1))}{s-1} \frac{2i\pi(s-1)}{\exp(2i\pi(s-1)) - 1} \\
&= \frac{1}{s-1} \left( \sum_{k \geq 0} i^k \pi^k \frac{1}{k!} (s-1)^k \right) \left( \sum_{k \geq 0} i^k (2\pi)^k \frac{B_k}{k!} (s-1)^k \right) \\
&= \frac{1}{s-1} + \sum_{k \geq 1} (-1)^k \left( \sum_{j=0}^{2k} \binom{2k}{j} 2^j B_j \right) \frac{\pi^{2k}}{(2k)!} (s-1)^{2k-1},
\end{aligned}$$

where  $B_j$  are the Bernoulli numbers. Euler's identity:

$$\zeta(2k) = (-1)^{k+1} 2^{2k-1} B_{2k} \frac{\pi^{2k}}{(2k)!} \quad (k \geq 1),$$

then allows us to rewrite this expansion as follows:

$$\frac{-\pi}{\sin(\pi s)} = \frac{1}{s-1} - \sum_{k \geq 1} \frac{2^{1-2k}}{B_{2k}} \sum_{j=0}^{2k} \binom{2k}{j} 2^j B_j \zeta(2k) (s-1)^{2k-1}.$$

Moreover, the latter expression can be simplified thanks to the identity

$$\sum_{j=0}^k \binom{k}{j} 2^j B_j = 2^k B_k \left(\frac{1}{2}\right) = 2(1 - 2^{k-1}) B_k \quad (k \geq 2).$$

Hence, the expansion of  $\frac{\pi}{\sin(\pi s)}$  at  $s = 1$  is given by

$$\frac{\pi}{\sin(\pi s)} = -\frac{1}{s-1} - \sum_{k=1}^{\infty} \frac{2^{2k-1} - 1}{2^{2k-2}} \zeta(2k) (s-1)^{2k-1}.$$

On the other hand, the expansion of  $\zeta(s)$  at  $s = 1$  is

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k,$$

where  $\gamma_k$  are the Stieltjes constants. The expansion of  $\frac{\pi}{\sin(\pi s)} \zeta(s)$  is then obtained by Cauchy product. We have

$$\begin{aligned}
\frac{\pi}{\sin(\pi s)} \zeta(s) &= -\frac{1}{(s-1)^2} - \frac{\gamma}{(s-1)} + \gamma_1 - \zeta(2) \\
&\quad - \left( \frac{1}{2} \gamma_2 + \gamma \zeta(2) \right) (s-1) + \left( \frac{1}{6} \gamma_3 + \gamma_1 \zeta(2) - \frac{7}{4} \zeta(4) \right) (s-1)^2 \\
&\quad - \left( \frac{1}{24} \gamma_4 + \frac{1}{2} \gamma_2 \zeta(2) + \frac{7}{4} \gamma \zeta(4) \right) (s-1)^3 \\
&\quad + \left( \frac{1}{120} \gamma_5 + \frac{1}{6} \gamma_3 \zeta(2) + \frac{7}{4} \gamma_1 \zeta(4) - \frac{31}{16} \zeta(6) \right) (s-1)^4 - \dots \quad (6)
\end{aligned}$$

In (6), the coefficient of  $(s-1)^{2k}$  and  $(s-1)^{2k-1}$  are respectively

$$\frac{\gamma_{2k+1}}{(2k+1)!} + \sum_{j=1}^k \frac{\gamma_{2k-2j+1}}{(2k-2j+1)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \zeta(2j) - \frac{(2^{2k+1}-1)}{2^{2k}} \zeta(2k+2),$$

and

$$-\frac{\gamma_{2k}}{(2k)!} - \sum_{j=1}^k \frac{\gamma_{2k-2j}}{(2k-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}} \zeta(2j).$$

b) The well-known Taylor series expansion of the digamma function:

$$\psi(x+1) + \gamma = \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(n+1) x^n \quad (|x| < 1),$$

allows us write

$$\int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n-(s-1)}.$$

This leads to the following expansion:

$$\begin{aligned} \int_0^1 \frac{\psi(x+1) + \gamma}{x^s} dx &= \sum_{k=0}^{\infty} \left( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n^{k+1}} \right) (s-1)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times k! \tau_{k+1} (s-1)^k \quad (|s-1| < 1). \end{aligned} \quad (7)$$

c) To obtain the expansion of  $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s}$  at  $s=1$ , we proceed as follows: first we write the identities

$$\frac{H_n}{n^s} = \frac{H_n}{n} e^{-(s-1)\log n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{H_n (\log n)^k}{n} (s-1)^k,$$

so that

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{n \geq 1}^{\mathcal{R}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{H_n (\log n)^k}{n} (s-1)^k \right).$$

Moreover, [9, Theorem 9] allows us interchange  $\sum_{n \geq 1}^{\mathcal{R}}$  and  $\sum_{k=0}^{\infty}$ . This leads to the expansion:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n (\log n)^k}{n} \right) (s-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} r_k (s-1)^k. \quad (8)$$

The desired formulas (3) and (4) can then be easily obtained by combining the previous expansions (6), (7), and (8).  $\square$

### 3 New expression of $\zeta_H(s)$ and its consequences

In this section, we give a new expression of the harmonic zeta function (Theorem 1) from which follows a new evaluation of the harmonic Stieltjes constants (Corollary 1). Another interesting consequence of this new representation of  $\zeta_H$  is that it allows us to derive a new integral formula for  $\zeta(n)$  (Corollaries 2 and 3), as well as new evaluations of Glasser-Manna type integrals (Corollaries 4–6).

**Theorem 1.** *For all complex numbers  $s$  in  $\mathbb{C} \setminus \mathbb{Z}$ , we have*

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\Phi(s) \quad (9)$$

with

$$\Phi(s) := \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left( \text{Log}(1 + e^{ix}) \right)^{s-1} dx, \quad (10)$$

where  $\text{Log}$  denotes the principal value of the complex logarithm.

*Proof.* Let  $z = \text{Log}(1+e^{ix}) = \log\left(2 \cos\left(\frac{x}{2}\right)\right) + \frac{ix}{2}$ ,  $-\pi < x < \pi$ . When  $x$  varies from  $-\pi$  to  $\pi$ , the variable  $z$  traverses the path  $L$  defined by the parametric equations  $\text{Re } z = \log(2 \cos(x/2))$  and  $\text{Im } z = x/2$ . This path extends from  $-\infty$  below the line  $\text{Im } z = 0$ , passes through the point  $(\log 2, 0)$ , then extends back to  $-\infty$  above the line  $\text{Im } z = 0$ . Since  $L$  is homotopic to the Hankel contour  $C$  defined in [6, 7], then by means of [7, Equation (25)], we have, for  $s \in \mathbb{C} \setminus \mathbb{Z}$ , the representation

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + \Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s), \quad (11)$$

where

$$\phi(s) := \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}\left(\frac{e^z - 1}{z}\right) dz.$$

We now show that the equations (9) and (11) are equivalent. To do this, we proceed as follows. Differentiation of the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z}{1 - e^z} dz$$

[6, Equation (2.4)], leads to the following identity:

$$\psi(1-s)\zeta(s) + \zeta'(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1 - e^z} dz \quad (s \neq 1, 2, 3, \dots) \quad (12)$$

[6, Equation (2.7)]. Splitting the right-hand member of (12), we can write

$$\frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z \text{Log}(z)}{1 - e^z} dz = \phi(s) - \Phi(s), \quad (13)$$



with

$$\Phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \text{Log}(e^z - 1) dz.$$

From (12) and (13), we deduce the identity

$$\Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s) = \Gamma(1-s)\Phi(s).$$

Moreover, the substitution

$$z = \text{Log}(1 + e^{ix}), \quad -\pi < x < \pi,$$

in the contour integral defining  $\Phi$  gives this function the equivalent form (10), proving that (9) and (11) are equivalent equations.  $\square$

### 3.1 New formulas for $\gamma^{(k)}$

We deduce from (9) an expression of  $\gamma^{(k)}$  which seems to us more meaningful than the one given by Kargin et al. [16, Equation (21)].

**Corollary 1.** For any non-negative integer  $n$ , let  $\mathcal{L}_n$  and  $\xi_n$  be defined as follows:

$$\mathcal{L}_n := \Phi^{(n)}(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \text{Log}^n(\text{Log}(1 + e^{ix})) dx,$$

and

$$\xi_n := (-1)^n \Gamma^{(n)}(1) = (-1)^n \int_0^{+\infty} e^{-x} \log^n(x) dx.$$

Then we have

$$\gamma^{(0)} = -\gamma_1 - \zeta(2) - \mathcal{L}_1, \tag{14}$$

and, for all positive integers  $k$ , the following general relations hold true:

$$\begin{aligned} \gamma^{(2k-1)} = & -\zeta^{(2k-1)}(2) - \frac{\gamma_{2k}}{2k} + \frac{1}{2k} \sum_{n=0}^{2k-1} \binom{2k}{n} \xi_n \mathcal{L}_{2k-n} \\ & + \sum_{n=1}^k \frac{2(2k-1)!}{(2k-2n)!} \gamma_{2k-2n} \zeta(2n), \end{aligned} \tag{15}$$

and

$$\begin{aligned} \gamma^{(2k)} = & \zeta^{(2k)}(2) - \frac{\gamma_{2k+1}}{2k+1} - 2(2k)! \zeta(2k+2) - \frac{1}{2k+1} \sum_{n=0}^{2k} \binom{2k+1}{n} \xi_n \mathcal{L}_{2k+1-n} \\ & + \sum_{n=1}^k \frac{2(2k)!}{(2k+1-2n)!} \gamma_{2k+1-2n} \zeta(2n). \end{aligned} \tag{16}$$

Moreover, it follows from (1) and (14) the explicit evaluation:

$$\mathcal{L}_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \operatorname{Log}(\operatorname{Log}(1 + e^{ix})) dx = -\frac{3}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1. \quad (17)$$

**Example 2.** Applying formulas (15), (16) to the case  $k = 1$  and using (17) above, we then obtain

$$\gamma^{(1)} = -\zeta'(2) - \frac{1}{2}\gamma^3 - \frac{1}{2}\gamma_2 + \frac{1}{2}\mathcal{L}_2 + \frac{1}{2}\gamma\zeta(2) - \gamma\gamma_1,$$

and

$$\gamma^{(2)} = \zeta''(2) + \frac{1}{2}\gamma^4 - \frac{1}{3}\gamma_3 - \frac{1}{3}\mathcal{L}_3 - \gamma\mathcal{L}_2 + (2\gamma^2 + 5\gamma_1)\zeta(2) + \gamma^2\gamma_1 - \frac{1}{10}(\zeta(2))^2.$$

Numerical evaluations of the constants  $\mathcal{L}_2$  and  $\mathcal{L}_3$  are

$$\mathcal{L}_2 = -1.924491\dots \quad \text{and} \quad \mathcal{L}_3 = 7.158075\dots$$

*Proof of Corollary 1.* The key formula to derive the relations (14), (15) and (16) is the representation of  $\zeta_H(s)$  given by (9). Fortunately, the Laurent series expansion of each component in (9) can be written explicitly.

a) The Laurent expansions of  $\pi \cot(\pi s)$  and  $\zeta(s)$  at  $s = 1$  are respectively

$$\pi \cot(\pi s) = \frac{1}{s-1} - \sum_{k \geq 1} 2\zeta(2k)(s-1)^{2k-1},$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k \geq 1} \frac{(-1)^k}{k!} \gamma_k (s-1)^k,$$

where the coefficients  $\gamma_k$  are the Stieltjes constants. The expansion of  $\pi \cot(\pi s) \zeta(s)$  is then obtained by Cauchy product:

$$\begin{aligned} \pi \cot(\pi s) \zeta(s) &= \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 - 2\zeta(2) - \left(-\frac{1}{2}\gamma_2 + 2\gamma\zeta(2)\right)(s-1) \\ &\quad + \frac{1}{2} \left(-\frac{1}{3}\gamma_3 + 4\zeta(2)\gamma_1 - 4\zeta(4)\right)(s-1)^2 \\ &\quad - \frac{1}{6} \left(-\frac{1}{4}\gamma_4 + 6\zeta(2)\gamma_2 + 12\gamma\zeta(4)\right)(s-1)^3 + \dots \quad (18) \end{aligned}$$

b) It follows from the definition of  $\xi_k$  as  $(-1)^k \Gamma^{(k)}(1)$  that the Laurent expansion of  $\Gamma(1-s)$  at  $s = 1$  is given by

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \sum_{k=2}^{\infty} \frac{\xi_k}{k!} (s-1)^{k-1} \quad (|s-1| < 1),$$

On the other hand, the function  $\Phi$  defined by (10) is an entire function of  $s$  with  $\Phi(1) = \mathcal{L}_0 = 0$ , and the definition of  $\mathcal{L}_k$  as  $\Phi^{(k)}(1)$  implies that the Taylor expansion of  $\Phi(s)$  at  $s = 1$  is given by

$$\Phi(s) = \sum_{k=1}^{\infty} \frac{\mathcal{L}_k}{k!} (s-1)^k.$$

The Laurent expansion of  $\Gamma(1-s)\Phi(s)$  then follows by Cauchy product:

$$\begin{aligned} \Gamma(1-s)\Phi(s) &= -\mathcal{L}_1 - \left(\frac{1}{2}\mathcal{L}_2 + \gamma\mathcal{L}_1\right)(s-1) \\ &\quad + \frac{1}{2}\left(-\frac{1}{3}\mathcal{L}_3 - \gamma\mathcal{L}_2 - 2\gamma^{(0)}\mathcal{L}_1\right)(s-1)^2 \\ &\quad - \frac{1}{6}\left(\frac{1}{4}\mathcal{L}_4 + \gamma\mathcal{L}_3 + 3\gamma^{(0)}\mathcal{L}_2 + (2\zeta(3) + 3\gamma\zeta(2) + \gamma^3)\mathcal{L}_1\right)(s-1)^3 + \dots \end{aligned} \quad (19)$$

c) The Taylor expansion of  $\zeta(s+1)$  at  $s = 1$  can be written as follows:

$$\zeta(s+1) = \zeta(2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (-1)^k \zeta^{(k)}(2) (s-1)^k. \quad (20)$$

By assembling equations (18), (19) and (20) above, we obtain, by identifying the constant term, the relation (14). In the same way, the general formulae (15) and (16) are derived by identifying the coefficients of higher degree.  $\square$

**Remark 2.** a) The constants  $\xi_n$  involved in formulae (15)–(16) have a closed form expression in terms of Euler's constant  $\gamma$  and  $\zeta(k)$  for  $2 \leq k \leq n$  [3, 11]. More precisely, we have  $\xi_0 = 1$ ,  $\xi_1 = \gamma$ , and the recursion formula [11, Equation (2.2)]

$$\xi_{n+1} = \gamma\xi_n + \sum_{k=1}^n \frac{n!}{(n-k)!} \zeta(k+1) \xi_{n-k} \quad (n \geq 1),$$

thus, we have

$$\xi_2 = \gamma^2 + \zeta(2), \quad \xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3), \quad \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3(\zeta(2))^2 + 6\zeta(4).$$

A clever reformulation of this result is the following:

$$\xi_n = P_n(\gamma, \zeta(2), \dots, \zeta(n)) \quad (n \geq 2),$$

where  $P_n$  is the polynomial defined by the exponential generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = \sum_{n=0}^{\infty} P_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!}.$$

b) By means of a theorem of Ramanujan [12, Theorem 4] whose proof is given in [5, p. 224], we have the following identity:

$$(-1)^n \zeta^{(n)}(2) = n! + \gamma_n + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_{k+n} \quad (n \geq 1).$$

### 3.2 New formula for $\zeta(n)$

When written in a neighborhood of  $s = n$  (for any integer  $n \geq 2$ ), the relation between  $\Phi(s)$  and  $\zeta_H(s)$  given by formula (9) allows us to derive the following integral representation of  $\zeta(n)$ :

**Corollary 2.** For all integers  $n \geq 2$ , we have

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \Phi(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left( \text{Log}(1 + e^{ix}) \right)^{n-1} dx. \quad (21)$$

*Proof of Corollary 2.* For  $n \geq 2$ , we can write

$$\begin{aligned} \pi \cot(\pi s) \zeta(s) &= \frac{\zeta(n)}{s-n} + \zeta'(n) + O(s-n), \\ \Gamma(1-s) &= \frac{(-1)^n}{(n-1)!} \left( \frac{1}{s-n} + (\gamma - H_{n-1}) \right) + O(s-n), \end{aligned}$$

and

$$\Gamma(1-s) \Phi(s) = \frac{(-1)^n}{(n-1)!} \frac{\Phi(n)}{s-n} + \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + O(s-n).$$

By applying formula (9) around  $s = n$ , we deduce the equation

$$\begin{aligned} \zeta_H(n) + O(s-n) &= \left( \zeta(n) + \frac{(-1)^n}{(n-1)!} \Phi(n) \right) \frac{1}{s-n} \\ &+ \frac{(-1)^n}{(n-1)!} (\Phi'(n) + \Phi(n)(\gamma - H_{n-1})) + \zeta'(n) + \zeta(n+1) + O(s-n). \end{aligned}$$

The identity

$$\zeta(n) + \frac{(-1)^n}{(n-1)!} \Phi(n) = 0$$

results from the cancellation of the coefficient of  $\frac{1}{s-n}$  in the right-hand side of this equation, thus proving formula (21).  $\square$

**Remark 3.** It should be noted that our formula (21) for  $\zeta(n)$  is similar to but simpler than the one given by [6, Equation (2.6)] which, after the substitution  $z = \text{Log}(1 + e^{ix})$ , translates into

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Log} \left( \text{Log}(1 + e^{ix}) \right) \left( \text{Log}(1 + e^{ix}) \right)^{n-1} dx.$$

**Corollary 3.** For all positive integers  $n$ , we have

$$\Phi(2n+1) = (2n)! \zeta(2n+1), \quad (22)$$

$$(-1)^n \Phi(2n) = \frac{B_{2n}}{4n} (2\pi)^{2n}. \quad (23)$$

**Example 3.** Applying the previous formulas to the case  $n = 1$  gives

$$\begin{aligned} \Phi(3) &= 2\zeta(3) = \zeta_H(2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \operatorname{Log}^2(1 + e^{ix}) dx, \\ -\Phi(2) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} ix \operatorname{Log}(1 + e^{ix}) dx = \frac{\pi^2}{6}, \end{aligned}$$

### 3.3 New formulas for Glasser-Manna type integrals

When written in a neighborhood of  $s = -n$  (for any non-negative integer  $n$ ), the relation between  $\Phi(s)$  and  $\zeta_H(s)$  given by formula (9) allows us to derive new formulas for Glasser-Manna type integrals.

**Corollary 4.** We have

$$\Phi(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1 + e^{ix})} dx = \frac{1}{2} \log(2\pi) - \frac{1}{2}\gamma + \frac{1}{2}, \quad (24)$$

and

$$\begin{aligned} \Phi'(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix \operatorname{Log}(\operatorname{Log}(1 + e^{ix}))}{\operatorname{Log}(1 + e^{ix})} dx \\ &= \frac{1}{2}\gamma_1 + \frac{1}{4} \log^2(2\pi) + \frac{1}{4}\gamma^2 - \frac{1}{2}\gamma - \frac{1}{2}\gamma \log(2\pi) - \frac{7}{8}\zeta(2) + \beta, \end{aligned} \quad (25)$$

where  $\beta$  is the linear coefficient in the Laurent expansion of  $\zeta_H$  at  $s = 0$ .

*Proof of Corollary 4.* Around 0, we have the decomposition given by (9):

$$\zeta_H(s) = \zeta(s+1) + \pi \cot(\pi s) \zeta(s) + \Gamma(1-s)\Phi(s),$$

and the expansion [10, Equation (8)]

$$\zeta_H(s) = \frac{1}{2s} + \frac{1}{2}\gamma + \frac{1}{2} + \beta s + O(s^2).$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s+1) &= \frac{1}{s} + \gamma - \gamma_1 s + O(s^2), \\ \pi \cot(\pi s) \zeta(s) &= -\frac{1}{2s} - \frac{1}{2} \log(2\pi) + \left(\frac{1}{2}\gamma_1 - \frac{1}{4} \log^2(2\pi) + \frac{1}{4}\gamma^2 + \frac{7}{8}\zeta(2)\right)s + O(s^2), \\ \Gamma(1-s)\Phi(s) &= \Phi(0) + (\Phi'(0) + \gamma\Phi(0))s + O(s^2). \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand side of (9), we deduce the equation

$$\frac{1}{2}\gamma + \frac{1}{2} = \gamma - \frac{1}{2}\log(2\pi) + \Phi(0),$$

which is equivalent to (24). In the same way, formula (25) is derived by identifying the linear coefficient in the expansion of the right-hand side of (9).  $\square$

**Remark 4.** Formula (24) appears in [15] in a slightly different but equivalent form [15, Proposition 3.4], due to the identities

$$\Phi(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix dx}{2 \log(2 \cos(\frac{x}{2})) + ix} = \frac{2}{\pi} \int_0^{\pi} \frac{x^2}{x^2 + 4 \log^2(2 \cos(\frac{x}{2}))} dx.$$

**Remark 5.** An evaluation of the constant  $\beta$  which appears in formula (25) can be easily derived from [17, Corollary 4.2]; more precisely, we have

$$\beta = 1 + \frac{1}{2}\gamma - \frac{1}{4}\gamma^2 - \gamma_1 + \frac{1}{4}\zeta(2) - \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^2} = 1.589935\dots,$$

where  $b_n$  are the Bernoulli numbers of the second kind defined by way of their generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n x^n \quad (|x| < 1).$$

**Corollary 5.** We have

$$\Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^2(1 + e^{ix})} dx = \log(\mathcal{A}) - \frac{1}{12}\gamma + \frac{7}{24}, \quad (26)$$

where  $\mathcal{A} = \exp(\frac{1}{12} - \zeta'(-1))$  is the Glaisher-Kinkelin constant and, more generally, for all integers  $n \geq 2$ , we have

$$(2n-1)! \Phi(1-2n) = \log(\mathcal{A}_n) - \frac{B_{2n}}{2n}\gamma + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2}, \quad (27)$$

with  $\log(\mathcal{A}_n) := \frac{H_{2n-1} B_{2n}}{2n} - \zeta'(1-2n)$ .<sup>1</sup>

**Example 4.**

$$\Phi(-3) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^4(1 + e^{ix})} dx = \frac{1}{6} \log(\mathcal{A}_2) + \frac{1}{720}\gamma + \frac{1}{320}.$$

<sup>1</sup>These numbers  $\mathcal{A}_n$  are called *generalized Glaisher-Kinkelin constants* (cf. [13]).

*Proof of Corollary 5.* Around  $s = -1$ , we have the expansion [10, Equation (9)]

$$\zeta_H(s) = -\frac{1}{12(s+1)} - \frac{1}{12}\gamma - \frac{1}{8} + O(s+1).$$

On the other side, we have

$$\begin{aligned}\zeta(s+1) &= -\frac{1}{2} + O(s+1), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-1) + O(s+1), \\ \Gamma(1-s)\Phi(s) &= \Phi(-1) + O(s+1).\end{aligned}$$

Formula (26) is then deduced by identifying the constant coefficient in the expansion of the right-hand member of (9). In the same way, from [10, Proposition 2], the Laurent expansion around  $s = 1 - 2n$  for  $n \geq 2$  is given by

$$\zeta_H(s) = \frac{\zeta(1-2n)}{s+2n-1} - \frac{B_{2n}}{2n}\gamma + \frac{H_{2n-1}B_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} + O(s+2n-1)$$

which, by the same method, allows us to deduce formula (27).  $\square$

**Corollary 6.** For all positive integers  $n$ , we have

$$(2n)! \Phi(-2n) = -\zeta'(-2n) + (2n+1) \frac{B_{2n}}{4n}, \quad (28)$$

from which follows the reflection formula:

$$2(2n)! \Phi(-2n) = (-1)^{n+1} (2\pi)^{-2n} \Phi(2n+1) + (2n+1) \frac{B_{2n}}{2n}. \quad (29)$$

**Example 5.**

$$2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^3(1+e^{ix})} dx = -\zeta'(-2n) + \frac{1}{8} = \frac{1}{4\pi^2} \zeta(3) + \frac{1}{8}. \quad (30)$$

*Proof of Corollary 6.* Around  $s = -2n$ , we have

$$\zeta_H(s) = -\frac{B_{2n}}{4n} + \frac{B_{2n}}{2} + O(s+2n).$$

On the other side, we have the expansions

$$\begin{aligned}\zeta(s+1) &= -\frac{B_{2n}}{2n} + O(s+2n), \\ \pi \cot(\pi s) \zeta(s) &= \zeta'(-2n) + O(s+2n), \\ \Gamma(1-s)\Phi(s) &= (2n)! \Phi(-2n) + O(s+2n).\end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (9), we obtain formula (28). Moreover, the well-known identity

$$-2\zeta'(-2n) = (-1)^{n+1} (2\pi)^{-2n} (2n)! \zeta(2n+1)$$

and the relation  $(2n)! \zeta(2n+1) = \Phi(2n+1)$  given by (22) enables to deduce formula (29) from (28).  $\square$

## Appendix: Evaluation of Glasser-Manna integrals with shifted Mascheroni series

For any natural number  $k$ , let us consider the series

$$\sigma_k := \sum_{n=k+1}^{\infty} \frac{|b_n|}{n-k},$$

where  $b_n$  are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \dots$$

For positive values of  $k$ , the series  $\sigma_k$  are called *shifted Mascheroni series* [14]. The nice identities  $\sigma_0 = \gamma$  and  $\sigma_1 = \frac{1}{2} \log(2\pi) - \frac{1}{2}\gamma - \frac{1}{2}$  are well-known [14, Proposition 2]. The first one is a classical result due to Mascheroni, and the second allows us to deduce from formula (24) the following evaluation:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1+e^{ix})} dx = \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2}{x^2 + \log^2(2\cos(x))} dx = \sigma_1 + 1 = 1.13033\dots \quad (\text{A1})$$

This formula was first given by Glasser and Manna [15, Proposition 3.3] in a slightly different but equivalent form. Moreover, thanks to the simplest special cases of the following formula arising from [14, Proposition 3]:

$$\sum_{k=1}^n (-1)^{n+k} k! S(n, k) \sigma_{k+1} = -\zeta'(-n) - \frac{B_{n+1}}{n+1} (\gamma + H_{n+1} - H_n) \quad (n \geq 1), \quad (\text{A2})$$

with  $S(n, k)$  denoting the Stirling numbers of the second kind, we can easily derive from formulas (26) and (30) above further evaluations of the same kind, such as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^2(1+e^{ix})} dx = \sigma_2 + \frac{5}{12} = 0.49232\dots \quad (\text{A3})$$

and

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^3(1+e^{ix})} dx = -\sigma_2 + 2\sigma_3 + \frac{1}{8} = 0.15544\dots \quad (\text{A4})$$

More generally, it is possible to deduce from (A2) the following identities that are nothing more than reinterpretations of formulas (27) and (28) in terms of shifted



Mascheroni series  $\sigma_k$ . These are

$$\begin{aligned} \frac{(2n-1)!}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^{2n}(1+e^{ix})} dx &= \sum_{k=1}^{2n-1} (-1)^{k+1} k! S(2n-1, k) \sigma_{k+1} \\ &+ \frac{B_{2n} H_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k B_{2n-k}}{(2n-k)^2} \quad (n \geq 2), \end{aligned} \quad (\text{A5})$$

and

$$\frac{(2n)!}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\text{Log}^{2n+1}(1+e^{ix})} dx = \sum_{k=1}^{2n} (-1)^k k! S(2n, k) \sigma_{k+1} + (2n+1) \frac{B_{2n}}{4n} \quad (n \geq 1). \quad (\text{A6})$$

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