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# A Complement to Laurent expansion of harmonic zeta functions

Marc-Antoine Coppo\* , Bernard Candelpergher

**Abstract** We complement an earlier article dedicated to harmonic zeta functions by outlining a method for obtaining closed-form expressions of the Laurent series coefficients of the harmonic zeta function  $\zeta_H$  about its pole at  $s = 1$ . These coefficients are named harmonic Stieltjes constants by analogy with the classical case.

## 1 Two representations of the harmonic zeta function

We recall that the harmonic zeta function  $\zeta_H$  (noted  $h$  in [2]) is defined by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \quad \text{for } \operatorname{Re}(s) > 1,$$

where  $H_n$  are the classical harmonic numbers

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

We consider the function  $F$  defined by the integral representation

$$F(s) := \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \operatorname{Log} \left( \frac{e^z - 1}{z} \right) dz,$$

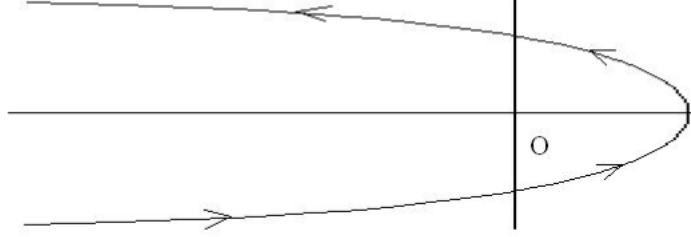
where  $\operatorname{Log}$  denotes the principal logarithm and  $L$  is the Hankel contour defined by the parametrization

$$z = \operatorname{Log}(1 + e^{ix}) \quad \text{with } x \in ] -\pi, \pi[ \quad \text{for all } z \in L.$$

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\*Corresponding author.

*E-mail address:* coppo@unice.fr (M-A. Coppo)



This function appears in [2] and plays a central role in the study of the function  $\zeta_H$  around its poles at the negative integers. The following representation:

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) - \psi(1-s)\zeta(s) - \zeta'(s) + F(s) \quad \text{for } s \text{ in } \mathbb{C} \setminus \mathbb{Z}, \quad (1)$$

where  $\psi(s) = \Gamma'(s)/\Gamma(s)$  is the digamma function, is a direct consequence of [2, Theorem 1] (see [2, Eqs. (8), (12) and (25)]).

On the other hand, differentiating the Hankel integral representation of  $\zeta$ :

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z}{1-e^z} dz$$

leads to the following identity:

$$\zeta'(s) + \psi(1-s)\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z \text{Log}(z)}{1-e^z} dz \quad \text{for } s \neq 1, 2, 3, \dots \quad (2)$$

(see [1, Eqs. (2.4)–(2.7)]). Furthermore, the above integral splits into two parts:

$$\int_L \frac{z^{s-1}e^z \text{Log}(z)}{1-e^z} dz = \int_L \frac{z^{s-1}e^z}{e^z-1} \text{Log}\left(\frac{e^z-1}{z}\right) dz - \int_L \frac{z^{s-1}e^z}{e^z-1} \text{Log}(e^z-1) dz,$$

allowing a rewriting of formula (2) as follows:

$$\zeta'(s) + \psi(1-s)\zeta(s) = F(s) - G(s), \quad (3)$$

with

$$G(s) := \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z}{e^z-1} \text{Log}(e^z-1) dz.$$

Finally, a substitution of (3) in formula (1) leads to another simpler representation of  $\zeta_H$  involving the function  $G$ . We have

$$\zeta_H(s) = \pi \cot(\pi s) \zeta(s) + \zeta(s+1) + G(s) \quad \text{for } s \text{ in } \mathbb{C} \setminus \mathbb{Z}. \quad (4)$$

## 2 Laurent series expansions at $s = 1$

The splitting of  $\zeta_H(s)$  into three parts given by formula (4) above is the key formula for obtaining the Laurent series expansion of the harmonic zeta function about its (double) pole at  $s = 1$ . To show this, we will make use of the following expansions:

- a) The Laurent expansions of  $\pi \cot(\pi s)$  and  $\zeta(s)$  at  $s = 1$  are known. They are respectively

$$\pi \cot(\pi s) = \frac{1}{s-1} - 2\zeta(2)(s-1) - 2\zeta(4)(s-1)^3 - \dots,$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \frac{1}{2}\gamma_2(s-1)^2 + \dots,$$

where  $\gamma_n$  are the classical Stieltjes constants. The expansion of  $\pi \cot(\pi s) \zeta(s)$  is then deduced by Cauchy's product as follows:

$$\pi \cot(\pi s) \zeta(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - 2\zeta(2) - \gamma_1 + \left( \frac{1}{2}\gamma_2 - 2\gamma\zeta(2) \right) (s-1) + \dots \quad (5)$$

- b) The Taylor series expansion of  $\zeta(s+1)$  at  $s = 1$  is given by

$$\zeta(s+1) = \zeta(2) + \zeta'(2)(s-1) + \frac{1}{2}\zeta''(2)(s-1)^2 + \dots \quad (6)$$

Moreover, we will use afterwards a well-known expression of  $\zeta'(2)$ :

$$\zeta'(2) = \zeta(2)\gamma + \zeta(2) \log(2\pi) - 2\pi^2 \log(A),$$

where  $A$  is the Glaisher-Kinkelin constant defined by

$$\log A = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n k \log k - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right\}.$$

- c) The Laurent expansion of  $\Gamma(z)$  at  $z = 0$  is given by

$$\Gamma(z) = \sum_{k \geq 0} P_k(-\gamma, \dots, (-1)^k \zeta(k)) z^{k-1},$$

where the polynomials  $P_k$  are the modified Bell polynomials defined by

$$\begin{aligned} \exp\left(\sum_{k \geq 1} x_k \frac{z^k}{k}\right) &= \sum_{k \geq 0} P_k(x_1, \dots, x_k) z^k \\ &= 1 + x_1 z + \left(\frac{1}{2}x_2 + \frac{1}{2}x_1^2\right) z^2 + \left(\frac{1}{3}x_3 + \frac{1}{2}x_1 x_2 + \frac{1}{6}x_1^3\right) z^3 + \dots \end{aligned}$$

The Laurent expansion of  $\Gamma(1-s)$  at  $s=1$  is easily deduced from this expression by setting  $z=1-s$ . We have

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \left(\frac{1}{2}\zeta(2) + \frac{1}{2}\gamma^2\right)(s-1) - \left(\frac{1}{3}\zeta(3) + \frac{1}{2}\gamma\zeta(2) + \frac{1}{6}\gamma^3\right)(s-1)^2 + \dots \quad (7)$$

d) Let us consider the function

$$g(s) := G(s)/\Gamma(1-s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1}e^z}{e^z-1} \text{Log}(e^z-1) dz.$$

This is an entire function of  $s$ , and the change of variables

$$z = \text{Log}(1 + e^{ix}) \quad \text{with } x \in ]-\pi, \pi[$$

leads to the integral representation

$$g(s) = -\frac{1}{2i\pi} \int_{-\pi}^{\pi} x (\text{Log}(1 + e^{ix}))^{s-1} dx.$$

If we now define  $K_n$  by

$$K_n := g^{(n)}(1) = \frac{i\pi}{2} \int_{-1}^1 x \text{Log}^n(\text{Log}(1 + e^{i\pi x})) dx,$$

then  $K_0 = g(1) = 0$ , and the Taylor series expansion of  $g(s)$  at  $s=1$  is given by

$$g(s) = \sum_{n \geq 1} \frac{K_n}{n!} (s-1)^n. \quad (8)$$

e) The Laurent expansion of  $G(s)$  at  $s=1$  is deduced from (7) and (8) by Cauchy's product. The result is as follows:

$$G(s) = -K_1 - \left(\frac{K_2}{2} + \gamma K_1\right)(s-1) - \left(\frac{K_3}{6} + \gamma \frac{K_2}{2} + (\gamma^2 + \zeta(2)) \frac{K_1}{2}\right)(s-1)^2 + \dots \quad (9)$$

In particular, it results from the decomposition (4) and the expansions (5), (6), and (9) above that the constant term in the Laurent expansion of  $\zeta_H(s)$  at  $s=1$  is  $-\zeta(2) - \gamma_1 - K_1$ . However, by using yet another representation of  $\zeta_H$ , we have

shown (see [3, Corollary 1 and Lemma 2]) that this constant term is  $\frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)$ . By comparing these two expressions, we derive the relation

$$-\zeta(2) - \gamma_1 - K_1 = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)$$

which gives an expression of the integral  $K_1$  in terms of  $\zeta(2)$ ,  $\gamma$  and  $\gamma_1$ . We have

$$-K_1 = -\frac{i\pi}{2} \int_{-1}^1 x \operatorname{Log}(\operatorname{Log}(1 + e^{i\pi x})) dx = \frac{3}{2}\zeta(2) + \frac{\gamma^2}{2} + \gamma_1 = 2.561174\dots \quad (10)$$

*Remark 1.* Unfortunately, no such formula is known for the integral  $K_2$  whose decimal approximation is  $K_2 = -1.924491$ . More generally, it is not known whether the integrals  $K_n$  for  $n \geq 2$  may be expressed using classical constants.

### 3 Evaluation of the harmonic Stieltjes constants

The expansions (5), (6), and (9) above and the decomposition (4) enable us to write the Laurent series expansion of the harmonic zeta function  $\zeta_H$  about its double pole  $s = 1$ . We have

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \tilde{\gamma}_0 - \tilde{\gamma}_1(s-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \tilde{\gamma}_n (s-1)^n \quad (0 < |s-1| < 1),$$

where the coefficients  $\tilde{\gamma}_n$  are the *harmonic Stieltjes constants* which are so called by analogy with the classical Stieltjes constants (see [3, Remark 3]). Moreover, this provides a method to evaluate these constants. Thus, for the first two ones, we obtain the following closed-form expressions:

$$\tilde{\gamma}_0 = -\zeta(2) - \gamma_1 - K_1 = \frac{\gamma^2}{2} + \frac{\pi^2}{12} = 0,98905599\dots$$

and

$$\tilde{\gamma}_1 = \frac{1}{2}K_2 + 2\pi^2 \log A - \frac{1}{2}\gamma_2 - \frac{\pi^2}{12}\gamma - \frac{\pi^2}{6} \log 2\pi - \frac{1}{2}\gamma^3 - \gamma\gamma_1 = 0,40076\dots$$

*Remark 2.* The expressions for  $\tilde{\gamma}_0$  and  $\tilde{\gamma}_1$  given above coincide with the asymptotic formula for  $\tilde{\gamma}_n$  mentioned in [3] (see [3, Remark 3 c]). In this order, we have

$$\tilde{\gamma}_0 = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{H_k}{k} - \frac{1}{2} \log^2(n) - \gamma \log(n) \right\}$$

and

$$\tilde{\gamma}_1 = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{H_k}{k} \log k - \frac{1}{3} \log^3(n) - \frac{1}{2} \gamma \log^2(n) \right\}.$$

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