



HAL
open science

Remarks on a formula of Blagouchine

Marc-Antoine Coppo

► **To cite this version:**

| Marc-Antoine Coppo. Remarks on a formula of Blagouchine. 2021. hal-03197403v7

HAL Id: hal-03197403

<https://univ-cotedazur.hal.science/hal-03197403v7>

Preprint submitted on 10 Jun 2021 (v7), last revised 3 Jun 2024 (v23)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Remarks on a formula of Blagouchine

Marc-Antoine Coppo*

Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Abstract We make some comments on an amazing formula recently discovered by Blagouchine.

Keywords Complex integration, generalized Glaisher-Kinkelin constants, infinite series with zeta values.

1 Introduction

The aim of this short note is to emphasize the link between the sequence of integrals $\{\mathcal{I}_k\}_{k \geq 0}$ defined by

$$\mathcal{I}_k = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(2k + 1 + 2ix) \cosh(\pi x)} dx$$

and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [6, 9, 10]. In order to do this, we make use of an amazing formula recently discovered by Blagouchine [2, Theorem 2] that we combine with another nice formula given by the author in a previous article ([7, Proposition 1]). Let us note in passing that the following special case of Blagouchine's formula:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n} = \mathcal{I}_0$$

has already been mentioned (without proof) on page 1836 of [7].

*Corresponding author. *Email address:* coppo@unice.fr

2 Blagouchine's integral

First of all, we provide a complete simple proof of [2, Theorem 2] using Cauchy's residue theorem.

Proposition 1. For any integer $k \geq 0$, let \mathcal{I}_k be the complex valued integral

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(2k + 1 + 2ix) \cosh(\pi x)} dx$$

and μ_k be the infinite sum

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n+k}.$$

Then we have the identity

$$\mathcal{I}_k = \mu_k. \quad (1)$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2} + iz)}{(\frac{1}{2} + k + iz) \cosh(\pi z)}.$$

We have $\cosh(\pi z) = 0$ if and only if $z = i/2 + in$ with $n \in \mathbb{Z}$. For $n \geq 1$, the residue of f_k at $z = i/2 - in$ is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2} - n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi}.$$

We integrate on a closed contour composed of the interval $D_R = [-R, R]$ and the lower semicircle C_R of radius R with center at 0. By the residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi} \int_{C_R} f_k(z) dz + \frac{1}{2i\pi} \int_{D_R} f_k(z) dz = - \sum_{n=1}^{N_R} \text{Res}(f_k; \frac{i}{2} - in),$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z) dz + \int_{D_R} f_k(z) dz = 2 \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)}. \quad (2)$$

For $z \in C_R$, the parameterization $iz = Re^{it}$ with $-\pi/2 < t < \pi/2$, enables us to write

$$\begin{aligned} \left| \int_{C_R} f_k(z) dz \right| &= \left| \int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it}) \cosh(i\pi Re^{it})} Re^{it} dt \right| \\ &\leq \int_{-\pi/2}^{+\pi/2} \left| \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it}) \cosh(i\pi Re^{it})} \right| R dt. \end{aligned}$$

Since $\frac{3}{2} + Re^{it}$ is in the half-plane $\operatorname{Re}(z) > 3/2$, its absolute value is bounded by $\zeta(\frac{3}{2})$, i.e.

$$\left| \zeta\left(\frac{3}{2} + Re^{it}\right) \right| \leq \zeta\left(\frac{3}{2}\right).$$

Hence, when R increases towards infinity, we have the following limits:

$$\lim_{R \rightarrow \infty} \int_{C_R} f_k(z) dz = 0,$$

$$\lim_{R \rightarrow \infty} \int_{D_R} f_k(z) dz = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{\left(\frac{1}{2} + k + ix\right) \cosh(\pi x)} dx = 2\mathcal{I}_k,$$

and

$$\lim_{R \rightarrow \infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \mu_k.$$

This allows us to deduce formula (1) by passing to the limit in (2). \square

Remark 1. The constant

$$\mu_0 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\dots$$

has been thoroughly studied by Boyadzhiev [4] (see also [6, p. 142]). This constant is noted M in [4], K in [6], and also appears as ν_{-1} in [7]. By a well-known series representation of Euler's constant γ , we also have

$$\mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) \right) = \gamma = 0.577215\dots$$

Example 1. For $k = 0$ and $k = 1$ respectively, formula (1) translates into

$$\mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(1 + 2ix) \cosh(\pi x)} dx = \int_0^1 \frac{\psi(x+1) + \gamma}{x} dx, \quad (3)$$

where ψ is the digamma function, and

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(3 + 2ix) \cosh(\pi x)} dx = \gamma = -\psi(1). \quad (4)$$

3 Link with the generalized Glaisher-Kinkelin constants

Definition 1 ([1, 9, 10]). For any integer $k \geq 0$, the constant A_k is usually defined by

$$\ln(A_k) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^k \ln n - P_k(N) \right\},$$

where $P_k(N)$ is given by $P_0(N) = (N + \frac{1}{2}) \ln N - N$, and

$$P_k(N) = \left(\frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^k \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \right) \ln N - \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^k \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left\{ (1 - \delta_{k,j}) \sum_{i=1}^j \frac{1}{k-i+1} \right\} \quad (k \geq 1),$$

where B_j is the j -th Bernoulli number and $\delta_{k,j}$ is the Kronecker delta function. The numbers A_k (for $k = 0, 1, 2, \dots$) are the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). In particular, it follows from this definition that

$$\ln(A_0) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \ln n - \left(N + \frac{1}{2} \right) \ln N + N \right\},$$

and

$$\ln(A_1) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n \ln n - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right\}.$$

Remark 2. Adamchik [1, Proposition 4] has given a nice expression of these constants in terms of the derivatives of the Riemann zeta function. More precisely, he showed that

$$A_k = \exp \left\{ \frac{H_k B_{k+1}}{k+1} - \zeta'(-k) \right\} \quad (k \geq 0), \quad (5)$$

where $H_k = \sum_{j=1}^k \frac{1}{j}$ is the k -th harmonic number with the usual convention $H_0 = 0$.

The following relations are easily deduced by differentiation of Riemann's functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \quad (k \geq 1),$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (H_{2k-1} - \gamma - \ln 2\pi) \quad (k \geq 1).$$

This enable to deduce from Adamchik's formula (5) the expressions

$$A_{2k-1} = \exp \left\{ (-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (\gamma + \ln 2\pi) \right\} \quad (k \geq 1), \quad (6)$$

and

$$A_{2k} = \exp \left\{ (-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \right\} \quad (k \geq 1). \quad (7)$$

Example 2. The constant $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$ is the Stirling constant,

$$A_1 = \exp \left(\frac{1}{12} - \zeta'(-1) \right) = \exp \left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi) + \gamma}{12} \right)$$

is the Glaisher-Kinkelin constant, and

$$A_2 = \exp(-\zeta'(-2)) = \exp \left(\frac{\zeta(3)}{4\pi^2} \right).$$

Remark 3. Bendersky [3] introduced for the first time the sequence of numbers $L_k := \ln(A_k)$ without any consideration of their relation with the ζ -function. From the point of view of the summation of divergent series, the constants L_k should be interpreted as follows: if $\sum_{n \geq 1}^{\mathcal{R}} n^k \ln n$ denotes the \mathcal{R} -sum (i.e. the sum in the sense of Ramanujan's summation method [5]) of the divergent series $\sum_{n \geq 1} n^k \ln n$, then, for any integer $k \geq 0$, we have

$$\sum_{n \geq 1}^{\mathcal{R}} n^k \ln n = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln \Gamma_k(x+1) dx, \quad (8)$$

where Γ_k is the Bendersky generalized gamma function [3]. In particular, this function verifies

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \quad \text{for any integer } n \geq 1.$$

Unaware of Bendersky's work, Kurokawa and Ochiai [8, Theorem 2] have given a deep expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s, x)$ at $s = -k$. Precisely, they showed that

$$\Gamma_k(x) = \exp \{ \zeta'(-k, x) - \zeta'(-k) \} \quad \text{for } x > 0.$$

For the first values of k , formula (8) translates into the identities

$$\begin{aligned}\sum_{n \geq 1}^{\mathcal{R}} \ln n &= \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln \Gamma(x+1) dx, \\ \sum_{n \geq 1}^{\mathcal{R}} n \ln n &= \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x+1) dx,\end{aligned}$$

where $K = \Gamma_1$ is the Kinkelin hyperfactorial function.

Proposition 2. For any integer $k \geq 1$, we have the following identity:

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j). \quad (9)$$

Proof. We have shown [7, Proposition 1] that

$$\mu_{k+1} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1} H_j}{j+1} \quad (k \geq 1).$$

By means of (5), this expression may be rewritten as follows:

$$\mu_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) \quad (k \geq 1).$$

Hence, formula (9) results from (1). \square

Example 3. For $k = 2, 3, 4$, formula (9) translates into the following identities:

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(5 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + 1 - \frac{1}{2} \ln(2\pi) \quad (10)$$

$$\mathcal{I}_3 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(7 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{2} - \frac{1}{3} \ln(2\pi) - \frac{\zeta'(2)}{\pi^2} \quad (11)$$

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(9 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{3} - \frac{1}{4} \ln(2\pi) - \frac{3\zeta'(2)}{2\pi^2} - \frac{3\zeta(3)}{4\pi^2} \quad (12)$$

More generally, we obtain the following expression of \mathcal{I}_k which results from (9) and formulas (6)–(7):

Corollary 1. For any integer $k \geq 4$, we have

$$\begin{aligned}\mathcal{I}_k &= \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(2k+1 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{k-1} - \frac{1}{k} \ln(2\pi) \\ &+ \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k-1}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} (-1)^j \binom{k-1}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1).\end{aligned} \quad (13)$$

Remark 4. Blagouchine [2, Theorem 1] also established the relation

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix) \cosh(\pi x)} dx \quad (k \geq 0),$$

which allows us to deduce from (9) the following identity:

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) = \frac{1}{k} + \frac{1}{(k+1)^2} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix) \cosh(\pi x)} dx \quad (k \geq 1). \quad (14)$$

In the simplest case, this formula reduces to

$$\ln(\sqrt{2\pi}) = \frac{5}{4} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(3+2ix) \cosh(\pi x)} dx.$$

References

- [1] V. Adamchik, Polygamma functions of negative order, *J. Comput. Appl. Math.* **100** (1998), 191–199.
- [2] I. V. Blagouchine, A complement to a recent paper on some infinite sums with the zeta values, preprint, 2020. Available at <https://arxiv.org/abs/2001.00108>
- [3] L. Bendersky, Sur la fonction gamma généralisée, *Acta Math.* **61** (1933), 263–322.
- [4] K. N. Boyadzhiev, A special constant and series with zeta values and harmonic numbers, *Gazeta Matematica* **115** (2018), 1–16.
- [5] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.
- [6] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Math., vol. 240, Springer, 2007.
- [7] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. Appl.* **475** (2019), 1831–1841.
- [8] N. Kurokawa and H. Ochiai, Generalized Kinkelin’s formulas *Kodai Math. J.* **30** (2007), 195–212.
- [9] M. Perkins and R. A. Van Gorder, Closed-form calculation of infinite products of Glaisher-type related to Dirichlet series, *Ramanujan J.* **49** (2019), 371–389.
- [10] W. Wang, Some asymptotic expansions on hyperfactorial functions and generalized Glaisher-Kinkelin constants, *Ramanujan J.* **43** (2017), 513–533.