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# Remarks on a formula of Blagouchine

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**Abstract** We make some comments on an amazing formula recently discovered by Blagouchine.

**Keywords** Complex integration, generalized Glaisher-Kinkelin constants, infinite series with zeta values.

## 1 Introduction

The aim of this short note is to emphasize the link between the sequence of integrals  $\{\mathcal{I}_k\}_{k \geq 0}$  defined by

$$\mathcal{I}_k = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(2k + 1 + 2ix) \cosh(\pi x)} dx$$

and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [6, 9, 10]. In order to do this, we make use of an amazing formula recently discovered by Blagouchine [2, Theorem 2] that we combine with another nice formula given by the author in a previous article ([7, Proposition 1]). Let us note in passing that the following special case of Blagouchine's formula:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n} = \mathcal{I}_0$$

has already been mentioned (without proof) on page 1836 of [7].

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## 2 Blagouchine's integral

First of all, we provide a complete simple proof of [2, Theorem 2] using Cauchy's residue theorem.

**Proposition 1.** For any integer  $k \geq 0$ , let  $\mathcal{I}_k$  be the complex valued integral

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(2k + 1 + 2ix) \cosh(\pi x)} dx$$

and  $\mu_k$  be the infinite sum

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n+k}.$$

Then we have the identity

$$\mathcal{I}_k = \mu_k. \quad (1)$$

*Proof.* For  $k \geq 0$ , let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2} + iz)}{(\frac{1}{2} + k + iz) \cosh(\pi z)}.$$

We have  $\cosh(\pi z) = 0$  if and only if  $z = i/2 + in$  with  $n \in \mathbb{Z}$ . For  $n \geq 1$ , the residue of  $f_k$  at  $z = i/2 - in$  is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2} - n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi}.$$

We integrate on a closed contour composed of the interval  $D_R = [-R, R]$  and the lower semicircle  $C_R$  of radius  $R$  with center at 0. By the residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi} \int_{C_R} f_k(z) dz + \frac{1}{2i\pi} \int_{D_R} f_k(z) dz = - \sum_{n=1}^{N_R} \text{Res}(f_k; \frac{i}{2} - in),$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z) dz + \int_{D_R} f_k(z) dz = 2 \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)}. \quad (2)$$

For  $z \in C_R$ , the parameterization  $iz = Re^{it}$  with  $-\pi/2 < t < \pi/2$ , enables us to write

$$\begin{aligned} \left| \int_{C_R} f_k(z) dz \right| &= \left| \int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it}) \cosh(i\pi Re^{it})} Re^{it} dt \right| \\ &\leq \int_{-\pi/2}^{+\pi/2} \left| \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it}) \cosh(i\pi Re^{it})} \right| R dt. \end{aligned}$$

Since  $\frac{3}{2} + Re^{it}$  is in the half-plane  $\operatorname{Re}(z) > 3/2$ , its absolute value is bounded by  $\zeta(\frac{3}{2})$ , i.e.

$$\left| \zeta\left(\frac{3}{2} + Re^{it}\right) \right| \leq \zeta\left(\frac{3}{2}\right).$$

Hence, when  $R$  increases towards infinity, we have the following limits:

$$\lim_{R \rightarrow \infty} \int_{C_R} f_k(z) dz = 0,$$

$$\lim_{R \rightarrow \infty} \int_{D_R} f_k(z) dz = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{\left(\frac{1}{2} + k + ix\right) \cosh(\pi x)} dx = 2\mathcal{I}_k,$$

and

$$\lim_{R \rightarrow \infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \mu_k.$$

This allows us to deduce formula (1) by passing to the limit in (2).  $\square$

*Remark 1.* The constant

$$\mu_0 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\dots$$

has been thoroughly studied by Boyadzhiev [4] (see also [6, p. 142]). This constant is noted  $M$  in [4],  $K$  in [6], and also appears as  $\nu_{-1}$  in [7]. By a well-known series representation of Euler's constant  $\gamma$ , we also have

$$\mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) \right) = \gamma = 0.577215\dots$$

**Example 1.** For  $k = 0$  and  $k = 1$  respectively, formula (1) translates into

$$\mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(1 + 2ix) \cosh(\pi x)} dx = \int_0^1 \frac{\psi(x+1) + \gamma}{x} dx, \quad (3)$$

where  $\psi$  is the digamma function, and

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(3 + 2ix) \cosh(\pi x)} dx = \gamma = -\psi(1). \quad (4)$$

### 3 Link with the generalized Glaisher-Kinkelin constants

**Definition 1** ([1, 9, 10]). For any integer  $k \geq 0$ , the constant  $A_k$  is usually defined by

$$\ln(A_k) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^k \ln n - P_k(N) \right\},$$

where  $P_k(N)$  is given by  $P_0(N) = (N + \frac{1}{2}) \ln N - N$ , and

$$P_k(N) = \left( \frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^k \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \right) \ln N - \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^k \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left\{ (1 - \delta_{k,j}) \sum_{i=1}^j \frac{1}{k-i+1} \right\} \quad (k \geq 1),$$

where  $B_j$  is the  $j$ -th Bernoulli number and  $\delta_{k,j}$  is the Kronecker delta function. The numbers  $A_k$  (for  $k = 0, 1, 2, \dots$ ) are the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). In particular, it follows from this definition that

$$\ln(A_0) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \ln n - \left( N + \frac{1}{2} \right) \ln N + N \right\},$$

and

$$\ln(A_1) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n \ln n - \left( \frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right\}.$$

*Remark 2.* Adamchik [1, Proposition 4] has given a nice expression of these constants in terms of the derivatives of the Riemann zeta function. More precisely, he showed that

$$A_k = \exp \left\{ \frac{H_k B_{k+1}}{k+1} - \zeta'(-k) \right\} \quad (k \geq 0), \quad (5)$$

where  $H_k = \sum_{j=1}^k \frac{1}{j}$  is the  $k$ -th harmonic number with the usual convention  $H_0 = 0$ .

The following relations which are easily deduced by differentiation of Riemann's functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \quad (k \geq 1),$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (H_{2k-1} - \gamma - \ln 2\pi) \quad (k \geq 1)$$

enable to deduce from Adamchik's formula (5) the expressions

$$A_{2k-1} = \exp \left\{ (-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (\gamma + \ln 2\pi) \right\} \quad (k \geq 1), \quad (6)$$

and

$$A_{2k} = \exp \left\{ (-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \right\} \quad (k \geq 1). \quad (7)$$

**Example 2.** The constant  $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$  is the Stirling constant,

$$A_1 = \exp \left( \frac{1}{12} - \zeta'(-1) \right) = \exp \left( -\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi) + \gamma}{12} \right)$$

is the Glaisher-Kinkelin constant, and

$$A_2 = \exp(-\zeta'(-2)) = \exp \left( \frac{\zeta(3)}{4\pi^2} \right).$$

*Remark 3.* Bendersky [3] introduced for the first time the sequence of numbers  $L_k := \ln(A_k)$  without any consideration of their relation with the  $\zeta$ -function. From the point of view of the summation of divergent series, the constants  $L_k$  should be interpreted as follows: if  $\sum_{n \geq 1}^{\mathcal{R}} n^k \ln n$  denotes the  $\mathcal{R}$ -sum (i.e. the sum in the sense of Ramanujan's summation method [5]) of the divergent series  $\sum_{n \geq 1} n^k \ln n$ , then, for any integer  $k \geq 0$ , we have

$$\sum_{n \geq 1}^{\mathcal{R}} n^k \ln n = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln \Gamma_k(x+1) dx, \quad (8)$$

where  $\Gamma_k$  is the Bendersky generalized gamma function [3]. Unaware of Bendersky's work, Kurokawa and Ochiai [8, Theorem 2] have given a deep expression of the function  $\Gamma_k$  in terms of the derivative of the Hurwitz zeta function  $\zeta(s, x)$  at  $s = -k$ . Precisely, they showed that

$$\Gamma_k(x) = \exp \{ \zeta'(-k, x) - \zeta'(-k) \} \quad \text{for } x > 0.$$

In particular, it follows from this expression that

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \quad \text{for any integer } n \geq 1.$$

For the first values of  $k$ , formula (8) translates into the identities

$$\begin{aligned}\sum_{n \geq 1}^{\mathcal{R}} \ln n &= \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln \Gamma(x+1) dx, \\ \sum_{n \geq 1}^{\mathcal{R}} n \ln n &= \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x+1) dx,\end{aligned}$$

where  $K = \Gamma_1$  is the Kinkelin hyperfactorial function.

**Proposition 2.** For any integer  $k \geq 1$ , we have the following identity:

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j). \quad (9)$$

*Proof.* We have shown [7, Proposition 1] that

$$\mu_{k+1} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1} H_j}{j+1} \quad (k \geq 1).$$

By means of (5), this expression may be rewritten as follows:

$$\mu_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) \quad (k \geq 1).$$

Hence, formula (9) results from (1).  $\square$

**Example 3.** For  $k = 2, 3, 4$ , formula (9) translates into the following identities:

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(5 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + 1 - \frac{1}{2} \ln(2\pi) \quad (10)$$

$$\mathcal{I}_3 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(7 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{2} - \frac{1}{3} \ln(2\pi) - \frac{\zeta'(2)}{\pi^2} \quad (11)$$

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(9 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{3} - \frac{1}{4} \ln(2\pi) - \frac{3\zeta'(2)}{2\pi^2} - \frac{3\zeta(3)}{4\pi^2} \quad (12)$$

More generally, we obtain the following expression of  $\mathcal{I}_k$  which results from (9) and formulas (6)–(7):

**Corollary 1.** For any integer  $k \geq 4$ , we have

$$\begin{aligned}\mathcal{I}_k &= \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(2k+1 + 2ix) \cosh(\pi x)} dx = \frac{1}{2}\gamma + \frac{1}{k-1} - \frac{1}{k} \ln(2\pi) \\ &+ \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k-1}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} (-1)^j \binom{k-1}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1).\end{aligned} \quad (13)$$

*Remark 4.* Blagouchine [2, Theorem 1] also established the relation

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix) \cosh(\pi x)} dx \quad (k \geq 0),$$

which allows us to deduce from (9) the following identity:

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) = \frac{1}{k} + \frac{1}{(k+1)^2} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix) \cosh(\pi x)} dx \quad (k \geq 1). \quad (14)$$

In the simplest case, this formula reduces to

$$\ln(\sqrt{2\pi}) = \frac{5}{4} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(3+2ix) \cosh(\pi x)} dx.$$

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