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Andreas H\"oring

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# ADJOINT $(1, 1)$ -CLASSES ON THREEFOLDS

ANDREAS HÖRING

ABSTRACT. We answer a question of Filip and Tosatti concerning a basepoint-free theorem for transcendental  $(1, 1)$ -classes on threefolds.

## 1. INTRODUCTION

In a recent preprint Filip and Tosatti proposed the following transcendental version of the basepoint-free theorem:

**1.1. Conjecture.** [FT17, Conj.1.2] *Let  $X$  be a compact Kähler manifold, and let  $\alpha$  be a nef  $(1, 1)$ -class on  $X$  such that  $\alpha - K_X$  is nef and big. Then the class  $\alpha$  is semiample: there exists a morphism with connected fibres  $\psi : X \rightarrow Z$  onto a normal compact Kähler space and a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = \psi^* \alpha_Z$ .*

If  $\alpha$  is the class of a  $\mathbb{Q}$ -divisor (and hence  $X$  is projective), this statement is equivalent to the basepoint-free theorem (cf. [BCHM10, Thm.3.9.1] for  $\mathbb{R}$ -divisors). Filip and Tosatti proved this conjecture for surfaces [FT17, Thm.1.3]. The aim of this note is to clarify the situation in dimension three:

**1.2. Theorem.** *Conjecture 1.1 holds if  $\dim X = 3$  and  $\alpha - K_X$  is a Kähler class.*

The assumption is slightly stronger than in the conjecture, but should be enough for potential applications to the Kähler-Ricci flow. Theorem 1.2 was shown by Tosatti and Zhang [TZ18, Thm.1.4] when the class  $\alpha$  is nef, but not big, so we only have to deal with the case where the morphism  $\psi$  is bimeromorphic. More precisely we prove the following:

**1.3. Theorem.** *Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with terminal singularities. Let  $\omega$  be a Kähler class on  $X$  such that  $\alpha := K_X + \omega$  is nef and big. Then there exists a bimeromorphic morphism  $\psi : X \rightarrow Z$  onto a normal compact Kähler space with isolated rational singularities and a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = \psi^* \alpha_Z$ .*

If the curves contracted by  $\psi$  define an extremal ray in the (generalised) Mori cone of  $X$ , this is the contraction theorem for Kähler threefolds (see Remark 3.5). In the setting of Theorem 1.3 we contract, more generally, an extremal face of the Mori cone. The obvious idea is to reduce to the case of extremal rays by running a MMP  $X \dashrightarrow Y$  where we contract only  $\alpha_\bullet$ -trivial  $K_\bullet$ -negative extremal rays. However  $Y$  is typically *not* the space we are looking for. Indeed if one of the steps of the MMP is a flip, the space  $Y$  can contain  $\alpha_Y$ -trivial curves that are  $K_Y$ -positive. These curves can not be contracted by the MMP, so we need some additional argument.

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At the moment we do not know how to prove Conjecture 1.1 if the class  $\alpha - K_X$  is merely nef and big. The main reason is that for crepant birational contractions the geometry of the exceptional locus is more complicated, so the strategy used for  $K_X$ -negative extremal rays in [HP16, CHP16] will not work. However it is easy to settle a special case:

**1.4. Proposition.** *Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with terminal singularities such that  $K_X \equiv 0$ . Let  $\alpha$  be a nef and big  $(1,1)$ -class on  $X$ . Then there exists a bimeromorphic morphism  $\psi : X \rightarrow Z$  onto a normal compact Kähler space with rational singularities and a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = \psi^* \alpha_Z$ .*

The proof is based on the observation that the decomposition theorem [CHP16, Dru18] essentially reduces the problem to the case of surfaces proven by Filip and Tosatti. In a similar spirit the Beauville-Bogomolov decomposition [Bea83b] reduces Conjecture 1.1 for manifolds with trivial canonical class to the case of Hyperkähler manifolds. This case is of independent interest (cf. [AV18]), we hope to come back to it in the future.

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## 2. DEFINITIONS AND TERMINOLOGY

We will use standard terminology of the minimal model program (MMP) as explained in [KM98] or [Deb01], see also [HP16, Sect.2-3] for minor modifications in the Kähler setting. For notions from analytic geometry we refer to [DP04], [Dem12].

Let  $X$  be a normal compact complex space with at most rational singularities. Suppose that  $X$  is in the Fujiki class, i.e.  $X$  is bimeromorphic to a compact Kähler manifold. A  $(1,1)$ -class on  $X$  is an element of  $H_{\text{BC}}^{1,1}(X)$ , the Bott-Chern group of  $(1,1)$ -currents that are locally  $\partial\bar{\partial}$ -exact (cf. [HP16, Sect.2] for details). Even if  $X$  is projective, the inclusion

$$\text{NS}(X) \otimes \mathbb{R} \subset H_{\text{BC}}^{1,1}(X)$$

is typically not an equality. However if  $H^2(X, \mathcal{O}_X) = 0$ , we can apply Kodaira's criterion to a desingularisation to see that  $X$  is projective and every  $(1,1)$ -class is an  $\mathbb{R}$ -divisor class.

Let  $\alpha \in H_{\text{BC}}^{1,1}(X)$  be a  $(1,1)$ -class. An irreducible curve  $C \subset X$  is  $\alpha$ -trivial if  $\alpha \cdot C = 0$ . The class  $\alpha$  is big if it contains a Kähler current, it is modified Kähler if there exists a modification  $\mu : X' \rightarrow X$  and a Kähler form  $\omega$  on  $X'$  such that  $[\mu_* \omega] = \alpha$ . A modified Kähler class is big, moreover for every hypersurface  $S \subset X$ , the restriction  $\alpha|_S$  is big: since the hypersurface  $S$  is not in the image of the  $\mu$ -exceptional locus, the restriction of the current  $\mu_* \omega$  to  $S$  is well-defined and a Kähler current.

Following [Pau98, Defn.3] one can define when a  $(1,1)$ -class is nef. A nef  $(1,1)$ -class is always nef in the algebraic sense, i.e. for every subvariety  $Z \subset X$  we have

$$Z \cdot \alpha^{\dim Z} \geq 0.$$

For a nef and big class  $\alpha$  on  $X$  one defines the Null locus

$$\text{Null}(\alpha) = \bigcup_{Z \cdot \alpha^{\dim Z} = 0} Z,$$

where the union runs over all positive-dimensional subvarieties. A priori the Null locus is a countable union of subvarieties, but by a theorem of Collins and Tosatti [CT15] (applied to the pull-back of the class  $\alpha$  to a desingularisation) we know that the Null locus has only finitely many irreducible components. Moreover we have the following :

**2.1. Lemma.** *Let  $X$  be a normal compact complex space with isolated rational singularities. Let  $\alpha \in H_{\text{BC}}^{1,1}(X)$  be a nef and big class such that  $Z \cdot \alpha^{\dim Z} > 0$  for every positive-dimensional subvariety. Then  $\alpha$  is a Kähler class, i.e. there exists a Kähler form  $\omega$  on  $X$  such that  $[\omega] = \alpha$ .*

*Proof.* Let  $\mu : X' \rightarrow X$  be a resolution of singularities. Then  $\mu^*\alpha$  is a nef and big class such that the Null locus coincides with the  $\mu$ -exceptional locus. By [CT15, Thm.1.1] this implies that the non-Kähler locus of  $\mu^*\alpha$  coincides with the  $\mu$ -exceptional locus. Combining Demailly's regularisation [Dem92] and [Bou04, Thm.3.17(ii)] this implies that there exists a Kähler current  $T$  in the class  $\mu^*\alpha$  which is singular exactly along the exceptional locus. The push-forward  $\mu_*T$  is thus a Kähler current in the class  $\alpha$  that is singular at most in the finitely many singular points of  $X$ . Using a regularized maximum, we obtain the smooth form  $\omega$  (cf. the proof of [FT17, Thm.1.3] for details).  $\square$

### 3. BIMEROMORPHIC GEOMETRY

The following lemma is a consequence of Araujo's description of the mobile cone [Ara10, Thm.1.3], the proof is part of the proof of [HP16, Prop.7.11].

**3.1. Lemma.** *Let  $F$  be a projective surface such that  $H^2(F, \mathcal{O}_F) = 0$ . Let  $\alpha_F$  be a nef (1,1)-class such that*

$$\alpha_F^2 = 0, \quad K_F \cdot \alpha_F < 0.$$

*Then  $F$  is covered by  $\alpha_F$ -trivial curves.*

We will need a slightly stronger version of [HP16, Prop.7.11]:

**3.2. Lemma.** *Let  $Y$  be a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with terminal singularities. Let  $\omega_Y$  be a modified Kähler class on  $Y$  such that  $\alpha_Y := K_Y + \omega_Y$  is nef and big. Let  $S \subset Y$  be an integral surface such that  $\alpha_Y^2 \cdot S = 0$ . Then  $S$  is Moishezon and covered by  $\alpha_Y$ -trivial curves.*

*Proof.* Let  $\pi : S' \rightarrow S$  be the composition of normalisation and minimal resolution of  $S$ , then we have

$$(1) \quad K_{S'} = \pi^*K_S - E$$

with  $E$  an effective  $\mathbb{Q}$ -divisor on  $S'$  (see [HP16, Sect.4.1]). Since  $S$  is a Kähler space, the surface  $S'$  is Kähler.

*1st case.* *The pull-back  $\pi^*\alpha_Y|_S$  is numerically trivial.* Then we have  $-\pi^*K_Y|_S = \pi^*\omega_Y|_S$ . Since  $\omega_Y$  is modified Kähler the restriction  $\omega_Y|_S$  is a big (1,1)-class. Thus  $-\pi^*K_Y|_S$  is a big line bundle on  $S'$  and  $S'$  is Moishezon. In particular  $S'$  (and hence  $S$ ) is covered by curves, they are all  $\alpha_Y|_S$ -trivial.

*2nd case.* *The pull-back  $\pi^*\alpha_Y|_S$  is not numerically trivial.* The nef and big class  $\alpha_Y$  defines an intersection form on  $N^1(Y)$  which, by the Hodge index theorem, has signature  $(1, d)$ . Since  $\alpha_Y^2 \cdot S = 0$  this implies that either  $\alpha_Y \cdot S = 0$  or

$\alpha_Y \cdot S^2 < 0$ . The former case is excluded since  $\pi^* \alpha_Y|_S$  is not numerically trivial. By the adjunction formula and using  $K_Y = \alpha_Y - \omega_Y$  we obtain

$$K_S \cdot \alpha_Y|_S = (K_Y + S) \cdot S \cdot \alpha_Y = \alpha_Y^2 \cdot S - \alpha_Y \cdot \omega_Y \cdot S + \alpha_Y \cdot S^2.$$

The first term is zero, the other terms are negative, so  $K_S \cdot \alpha_Y|_S < 0$ . By (1) this implies

$$K_{S'} \cdot \pi^* \alpha_Y|_S \leq K_S \cdot \alpha_Y|_S < 0.$$

Since  $\pi^* \alpha_Y|_S$  is a nef class, this shows that  $K_{S'}$  is not pseudoeffective. In particular we have  $H^2(S', \mathcal{O}_{S'}) = H^0(S', K_{S'}) = 0$ . By Kodaira's criterion  $S'$  is projective and we conclude with Lemma 3.1.  $\square$

**3.3. Proposition.** *Let  $Y$  be a normal compact Kähler threefold with isolated rational singularities. Let  $\alpha_Y$  be a  $(1,1)$ -class on  $Y$  that is nef and big. Suppose that every irreducible component of  $\text{Null}(\alpha_Y)$  has dimension one. Then there exists a bimeromorphic morphism  $\mu_Y : Y \rightarrow Z$  onto a normal compact complex space  $Z$  such that every connected component of  $\text{Null}(\alpha_Y)$  is contracted onto a point, and*

$$Y \setminus \text{Null}(\alpha_Y) \simeq Z \setminus \mu_Y(\text{Null}(\alpha_Y))$$

*is an isomorphism.*

**3.4. Remark.** We do not claim that  $\alpha_Y$  is a pull-back of a  $(1,1)$ -class from  $Z$ . In fact this already fails for nef and big divisors on projective surfaces.

*Proof.* The proof of [HP16, Thm.7.12] applies without changes. Indeed the first part of the proof consists in verifying that the nef supporting class is nef and big and has one-dimensional non-Kähler locus (which is exactly our assumption). The rest of the proof uses only this property.  $\square$

**3.5. Remark.** We need a technical remark concerning the use of MMP in our setting: let  $X$  be normal  $\mathbb{Q}$ -factorial compact Kähler threefold with terminal singularities. Suppose either that  $K_X$  is pseudoeffective (or equivalently, that  $X$  is not uniruled), or that the MRC fibration is an almost holomorphic map  $X \dashrightarrow B$  onto a compact Kähler surface. Let  $\alpha$  be a nef and big class on  $X$  and suppose that there exists a curve  $C \subset X$  such that

$$\alpha \cdot C = 0, \quad K_X \cdot C < 0.$$

Then there exists a contraction  $\mu : X \rightarrow X'$  of a  $K_X$ -negative extremal ray  $\Gamma$  such that  $\alpha \cdot \Gamma = 0$ .

If  $K_X$  is pseudoeffective, this is completely standard: by the cone theorem [HP16, Thm.1.2] we can decompose the class

$$C = \eta + \sum \lambda_i \Gamma_i$$

where  $K_X \cdot \eta \geq 0$ , the coefficients  $\lambda_i \geq 0$  and  $\Gamma_i$  are rational curves generating extremal rays in the cone  $\overline{\text{NA}}(X)$ . Since  $K_X \cdot C < 0$  there exists at least one coefficient  $\lambda_{i_0} > 0$ . Since  $\alpha$  is nef and  $\alpha \cdot C = 0$  we have  $\alpha \cdot \Gamma_{i_0} = 0$ . By the contraction theorem [HP16, Thm.1.3] we can contract the extremal ray  $\mathbb{R}^+ \Gamma_{i_0}$ .

If the base of the MRC fibration has dimension two, we only have weaker forms of the cone and contraction theorem [HP15]. Let  $F$  be a general fibre of the MRC-fibration. Then  $\alpha \cdot F > 0$ , since  $\alpha$  is nef and big. Let  $\omega_X$  be a Kähler class on  $X$ .

Then for all  $\varepsilon > 0$  the class

$$\omega_\varepsilon := 2 \frac{\alpha + \varepsilon(\alpha \cdot F)\omega_X}{(\alpha + \varepsilon(\alpha \cdot F)\omega_X) \cdot F}$$

is a normalised Kähler class in the sense of [HP15, Defn.1.2.]. Moreover, since  $\alpha \cdot C = 0$ , we know that for  $0 < \varepsilon \ll 1$  the intersection number

$$(K_X + \omega_\varepsilon) \cdot C < 0.$$

Thus we can use [HP15, Thm.3.13, Thm.3.15] to conclude as above.

*Proof of Theorem 1.3.* If  $H^2(X, \mathcal{O}_X) = 0$  the nef and big class  $\alpha$  is an  $\mathbb{R}$ -divisor class, so we can conclude with the basepoint-free theorem for  $\mathbb{R}$ -divisors. Suppose from now on that  $H^2(X, \mathcal{O}_X) \neq 0$ . Then  $X$  is either not uniruled or the MRC fibration is an almost holomorphic map  $X \dashrightarrow B$  onto a compact Kähler surface (cf. [HP15, Sect.1]). Note that these properties are invariant under the minimal model program.

*Step 1. Running a MMP.* We start by running an  $\alpha_\bullet$ -trivial  $K_\bullet$ -MMP. More precisely set

$$X_0 := X, \quad \omega_0 := \omega, \quad \alpha_0 := \alpha.$$

We define inductively a sequence of bimeromorphic maps as follows: for  $i \in \mathbb{N}$ , suppose that there exists a curve  $C \subset X$  such that  $K_{X_{i-1}} \cdot C < 0$  and  $\alpha \cdot C = 0$ . By Remark 3.5 there exists a contraction  $\varphi_i : X_{i-1} \rightarrow X_i$  of a  $K_{X_{i-1}}$ -negative extremal ray  $\Gamma_{i-1}$  such that  $\alpha_{i-1} \cdot \Gamma_{i-1} = 0$ . The contraction is not of fibre type, since otherwise  $X_{i-1}$  is covered by  $\alpha_{i-1}$ -trivial curves, contradicting the hypothesis that  $\alpha_{i-1}$  is nef and big. Thus the contraction is bimeromorphic, and we denote by  $\mu_i : X_{i-1} \dashrightarrow X_i$  the divisorial contraction or, for a small ray, its flip. Since the contraction is  $\alpha_{i-1}$ -trivial, the push-forward  $(\mu_i)_* \alpha_{i-1} =: \alpha_i$  is a nef and big  $(1, 1)$ -class  $\alpha_i$  on  $X_i$ . We also set  $(\mu_i)_* \omega_{i-1} =: \omega_i$ , so  $K_{X_i} + \omega_i = \alpha_i$ . The class  $\omega_i$  is not necessarily Kähler, but since  $(\mu_i)^{-1}$  does not contract any divisors, it is a modified Kähler class.

By Mori's termination of flips for terminal threefolds [Mor88], the MMP terminates after finitely many steps, so composing the  $\mu_i$  we obtain a bimeromorphic morphism

$$\mu : X \dashrightarrow Y$$

such that  $Y$  is a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with terminal singularities, the class  $\mu_* \alpha =: \alpha_Y$  is nef and big and  $\alpha_Y = K_Y + \omega_Y$  with  $\omega_Y$  a modified Kähler class. Since  $Y$  is the outcome of the  $\alpha_\bullet$ -trivial  $K_\bullet$ -MMP, we have  $\alpha_Y \cdot C > 0$  for every curve  $C$  such that  $K_Y \cdot C < 0$ .

We claim that  $\text{Null}(\alpha_Y)$  does not contain an irreducible surface  $S$ : otherwise we know by Lemma 3.2 that  $S$  is covered  $\alpha_Y$ -trivial negative curves  $(C_t)_{t \in T}$ . Since  $\omega_Y$  is modified Kähler, the restriction  $\omega_Y|_S$  is a big  $(1, 1)$ -class. Thus for a general curve  $C_t$  we have  $\omega_Y \cdot C_t > 0$ . Thus we obtain that  $K_Y \cdot C_t < 0$ , in contradiction to the preceding paragraph.

*Step 2. Construction of  $Z$ .* By the claim the Null locus of  $\alpha_Y$  has pure dimension one. By Proposition 3.3 there exists thus a bimeromorphic map  $\mu_Y : Y \rightarrow Z$  contracting the connected components of  $\text{Null}(\alpha_Y)$  onto points. We claim that the composition  $\psi := \mu_Y \circ \mu : X \dashrightarrow Z$  is holomorphic: denote by  $\Gamma_\mu \subset X \times Y$  the graph of  $\mu$  and by  $p : \Gamma_\mu \rightarrow X$  the projection onto  $X$ . Denote by  $\Gamma_\psi \subset X \times Z$

the graph of  $\psi$ . Then  $\psi$  is a morphism if and only if the projection  $\Gamma_\psi \rightarrow X$  is an isomorphism.

Since  $\psi$  is the composition of the meromorphic map  $\mu$  with the holomorphic map  $\mu_Y$  this is equivalent to showing that the  $p$ -fibres are contained in  $\text{Null}(\alpha_Y)$ <sup>1</sup>. Yet for a threefold-MMP, the  $p$ -fibres are the (strict transforms) of flipped curves. Since we ran an  $\alpha_\bullet$ -trivial MMP, the class  $\alpha_\bullet$  is trivial on both the contracted and the flipped curves. Thus the  $p$ -fibres are  $\alpha_Y$ -trivial.

Thus we have constructed a bimeromorphic morphism  $\psi : X \rightarrow Z$  such that the restriction of  $\alpha$  to every fibre is numerically trivial and the exceptional locus coincides with  $\text{Null}(\alpha)$ . Since  $\alpha = K_X + \omega$  and  $\omega$  is Kähler, this morphism is projective with relatively ample line bundle  $-K_X$ . In particular by relative Kodaira vanishing [Anc87] we have  $R^j \psi_* \mathcal{O}_X = 0$  for all  $j \geq 0$ . Since  $X$  has terminal, hence rational singularities, this implies that  $Z$  has rational singularities. By [HP16, Lemma 3.3.] there exists thus a (1,1)-class  $\alpha_Z$  on  $Z$  such that  $\alpha = \psi^* \alpha_Z$ . By Lemma 2.1 the class  $\alpha_Z$  is Kähler.  $\square$

**Remark.** In general the space  $Z$  is not  $\mathbb{Q}$ -factorial, since  $\mu_Y$  is a small contraction. Thus  $Z$  does not necessarily have terminal singularities.

#### 4. CALABI-YAU CASE

*Proof of Proposition 1.4.* By the decomposition theorem [Dru18, Thm.1.2] [CHP16, Thm.9.2] there exists a finite cover  $\mu : \tilde{X} \rightarrow X$  that is étale over the nonsingular locus such that  $\tilde{X}$  is either a torus, a Calabi-Yau threefold (in particular  $H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ ) or a product of an elliptic curve  $E$  with a K3 surface  $S$ .

If  $\tilde{X}$  is a torus, the pull-back  $\mu^* \alpha$  is Kähler. Thus  $\alpha$  itself is Kähler and we are done. If  $\tilde{X}$  is a Calabi-Yau, the (1,1)-class  $\alpha$  is an  $\mathbb{R}$ -divisor class and we conclude by the basepoint-free theorem.

Thus we are left to deal with the case  $\tilde{X} \simeq E \times S$ . Set  $\tilde{\alpha} := \mu^* \alpha$ . For a K3 surface we have  $H^0(S, \Omega_S) = 0$ , which immediately implies

$$H_{\text{BC}}^{1,1}(X) = p_E^* H_{\text{BC}}^{1,1}(E) \times p_S^* H_{\text{BC}}^{1,1}(S),$$

where  $p_E : \tilde{X} \rightarrow E$  and  $p_S : \tilde{X} \rightarrow S$  are the projections on the factors. Thus we can write

$$\tilde{\alpha} = \lambda F + p_S^* \alpha_S$$

where  $F$  is a  $p_E$ -fibre,  $\lambda \in \mathbb{R}$ , and  $\alpha_S$  is a (1,1)-class on  $S$ . The restriction of the nef and big class  $\tilde{\alpha}$  to a general  $p_E$ -fibre is nef and big, so  $\alpha_S$  is nef and big. The restriction of the nef and big class  $\tilde{\alpha}$  to a general  $p_S$ -fibre is nef and big, so  $\lambda > 0$ .

Let now  $Z$  be an irreducible component on the null locus of  $\tilde{\alpha}$ . If  $Z$  is a curve, we have

$$0 = \tilde{\alpha} \cdot Z = \lambda F \cdot Z + \alpha_S \cdot (p_S)_* Z.$$

By what precedes we obtain  $F \cdot Z = 0$ , so  $Z$  is contained in a fibre  $p_E^{-1}(t_0)$ . Yet then  $Z$  trivially deforms in a family  $Z_t = (t \times Z)_{t \in E}$ , and  $\alpha \cdot Z_t = 0$  for all the curves in this family. Thus  $Z$  is not an irreducible component of the null locus.

<sup>1</sup>Since  $\Gamma_\mu \subset X \times Y$ , the  $p$ -fibres are naturally embedded in  $Y$ .

This shows that any irreducible component of the null locus is a surface  $Z$ . If the map  $p_S|_Z : Z \rightarrow S$  is generically finite, the pull-back  $p_S^*|_Z \alpha_S$  is big. Hence  $\tilde{\alpha}|_Z$  is nef and big, but this contradicts the property of being in the null locus. Thus  $p_S(Z)$  is an irreducible curve, and since

$$Z \subset p_S^{-1}(p_S(Z)) = E \times p_S(Z)$$

is irreducible, we see that  $Z = E \times p_S(Z)$ . Since  $Z$  is in the null locus we have

$$0 = \tilde{\alpha}^2 \cdot Z = \lambda^2 F^2 \cdot Z + 2\lambda F \cdot p_S^* \alpha_S \cdot Z + (p_S^* \alpha_S)^2 \cdot Z = 2\lambda \alpha_S \cdot p_S(Z).$$

Since  $\lambda > 0$  we see that  $p_S(Z)$  is in the null-locus of  $\alpha_S$ , hence it is a  $(-2)$ -curve (cf. the proof of [FT17, Thm.1.3]). In conclusion we obtain that

$$\text{Null}(\tilde{\alpha}) = E \times \text{Null}(\alpha_S)$$

By [FT17, Thm.1.3] there exists a bimeromorphic map  $\psi_S : S \rightarrow S'$  onto a normal compact surface  $S'$  and a Kähler class  $\alpha_{S'}$  on  $S'$  such that  $\alpha_S = \psi_S^* \alpha_{S'}$ . We then set

$$\tilde{\psi} := id_E \times \psi_S : \tilde{X} = E \times S \rightarrow E \times S' =: \tilde{X}'$$

and  $\lambda F + \alpha_{S'}$  is a Kähler class on  $\tilde{X}'$  such that  $\tilde{\psi}^*(\lambda F + \alpha_{S'}) = \tilde{\alpha}$ .

Let us now show that this map descends to  $X$ : up to replacing  $\mu$  by its Galois closure, we have  $X = \tilde{X}/G$  where  $G$  is the Galois group of  $\mu$ . Any automorphism  $f$  on  $E \times S$  is of the form  $f_E \times f_S$  [Bea83a, p.8], thus  $G$  acts on the factors  $E$  and  $S$ . The class  $\mu^* \alpha$  is  $G$ -invariant, so  $\alpha_S$  is invariant under the  $G$ -action on  $S$ . The Null locus  $\text{Null}(\alpha_S)$  being  $G$ -invariant, we see that there is an induced  $G$ -action on  $S'$  that makes  $\psi_S$  is  $G$ -equivariant. Since  $\tilde{\psi} = id_E \times \psi_S$  there is an induced  $G$ -action on  $\tilde{X}'$  that makes  $\tilde{\psi}$  is  $G$ -equivariant. We set  $Z := \tilde{X}'/G$  and denote by  $\psi : X = \tilde{X}/G \rightarrow Z = \tilde{X}'/G$  the bimeromorphic morphism induced by  $\tilde{\psi}$ . Since  $\psi$  is crepant and  $X$  has terminal singularities, the space  $Z$  has canonical, hence rational singularities.

Denote by  $\mu' : \tilde{X}' \rightarrow Z = \tilde{X}'/G$  the finite cover. Since  $\mu^* \alpha$  is  $G$ -invariant, the Kähler class  $\lambda F + \alpha_{S'}$  is  $G$ -invariant and defines a Kähler class  $\alpha_Z$  on  $Z$ . By construction we have  $\alpha = \psi^* \alpha_Z$ .  $\square$

**Remark.** If  $\tilde{X} \simeq E \times S$ , then  $X$  is in general not a product. Let  $f_E$  be a fixpoint free involution (e.g. a translation by 2-torsion point), and  $f_S$  an involution with fixed points. Set  $X := \tilde{X}/\langle f_E \times f_S \rangle$  and denote by  $p : X \rightarrow S/\langle f_S \rangle$  the map induced by the projection  $p_S$ . Then  $p$  has multiple fibres over the fixed points, so  $X$  is not a product.

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ANDREAS HÖRING, UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE  
*E-mail address:* [Andreas.Hoering@unice.fr](mailto:Andreas.Hoering@unice.fr)