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# On certain alternating series involving zeta values

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## Introduction

This article is primarily devoted to the alternating series  $\nu_k$  defined by

$$\nu_k := \sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j+k},$$

where  $k$  denotes a complex parameter. By a classical result (cf. e.g. [Er], p. 45, Eq. (3) or [Jo], p. 62) which goes back to Euler's early works on zeta, one knows that  $\nu_0 = \gamma$ , where  $\gamma$  denotes the Euler-Mascheroni constant. It is less famous but yet fairly well-known (cf. [Sr], p. 135, Eq. (5.1), [SV], Eq. (1.5), [Ca], p. 93) that

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1.$$

In a recent paper, Blagouchine ([B1]) has obtained the following general expression for  $\nu_k$  in the case where  $k$  is a positive integer:

$$\begin{aligned} \nu_k &= \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} \\ &+ \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{\lfloor \frac{k+1}{2} \rfloor - 1} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1). \end{aligned} \quad (1)$$

However, this formula is quite cumbersome. Using the functional equation of zeta, we give an equivalent but simpler expression for this series in this case. We

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also examine the series  $\nu_k$  for complex values of the parameter  $k$ , a study which, as far as we know, have never been made before. Furthermore, we study certain natural generalizations of these series involving multiple zeta values.

In the case where  $k$  is a positive integer, we show that the series  $\nu_k$  admits the following explicit evaluation:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k, \quad (2)$$

where  $C_k$  is a rational constant for which we give two different, even though equivalent, expressions (cf. Proposition 1).

A further interesting generalization of the series  $\nu_k$  may be defined as follows: for any natural number  $p$ , we consider the series

$$\nu_{k,p} := \sum_{j=2}^{\infty} \frac{(-1)^j}{j+k} \zeta(j, \underbrace{1, \dots, 1}_p),$$

where

$$\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}},$$

in such a way that  $\nu_k = \nu_{k,0}$ . Then, for any integer  $k \geq -1$ , we show (cf. Proposition 2) the following identity:

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}. \quad (3)$$

Here, the numbers  $G_n^{(k)}$  are the *Gregory coefficients of higher order* introduced in [B2]. They may be either defined by

$$G_n^{(k)} := \frac{1}{n!} \sum_{m=1}^n \frac{S_1(n, m)}{m+k}, \quad \text{for } k = 0, 1, 2, \dots \quad (4)$$

where  $S_1(n, m)$  are the Stirling numbers of the first kind, or equivalently by the integral formula

$$G_n^{(k)} = \frac{(-1)^n}{n!} \int_0^1 x^{k-1} (-x)_n dx, \quad (5)$$

where  $(z)_n = z(z+1)(z+2) \dots (z+n-1)$  is the Pochhammer symbol. It follows from this last expression that  $G_n^{(k)} = (-1)^{n+1} |G_n^{(k)}|$ . Furthermore,  $G_n^{(1)} = G_n$ , where  $G_n$  are the *Gregory coefficients*, also known as the *Bernoulli numbers of the second kind*<sup>1</sup>.

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<sup>1</sup>Several authors quoted in reference use different notations for these numbers, they are noted  $b_n$  in [Jo], [CY], and  $\beta_n/n!$  in [Ca].

As a special case of the identity above, we deduce that

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n}, \quad \text{for } k = 0, 1, 2, \dots \quad (6)$$

and we note that this formula nicely generalizes the famous Mascheroni series for  $\gamma$  (which is nothing else than the case  $k = 1$ ).

Finally, the extension to the complex case is studied in the last section. We provide two beautiful integral formulas for the same series:

$$\nu_k = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(3/2 \pm ix + k) \cosh \pi x} dx, \quad (7)$$

and

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix + k) \cosh \pi x} dx, \quad (8)$$

which are valid for  $\text{Re}(k) > -\frac{3}{2}$  and  $\text{Re}(k) > -\frac{1}{2}$  respectively. The second representation seems especially interesting because the integral runs over the whole critical line.

In appendix, we return to the case of a positive integer  $k$  and highlight the existence of an interesting relation between the series  $\nu_k$ , the Stirling numbers of the second kind  $S_2(n, k)$  and the shifted Mascheroni series  $\sigma_r$  which have been recently studied in [CY].

## 1 The case of a positive integer

In this section, we study the case where the parameter  $k$  is a positive integer and give two independent proofs of our formula (2). More precisely, we prove the following proposition:

**Proposition 1.** For any positive integer  $k$ , then

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$\begin{aligned} C_k &= \frac{1}{k} + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2r} \frac{B_{2r} H_{2r-1}}{k+1-2r} \\ &= \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \frac{B_{2r}}{2r(k+1-2r)} \end{aligned} \quad (9)$$

where  $H_n$  are the harmonic numbers defined by

$$H_n = \sum_{m=1}^n \frac{1}{m}, \quad \text{for } n = 1, 2, 3, \dots$$

and  $B_n$  are the Bernoulli numbers defined by their exponential generating series

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}, \quad \text{for } |z| < 2\pi.$$

In particular,  $B_0 = 1$ ,  $B_1 = -1/2$ ,  $B_{2r+1} = 0$  for  $r \geq 1$ .

*Proof.* We can quite easily deduce (2) from formula (1). A differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1} \Gamma(1-s) \zeta(1-s) \sin \frac{\pi s}{2},$$

leads to the relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r), \quad \text{for } r = 1, 2, 3, \dots$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r} (H_{2r-1} - \gamma - \ln 2\pi), \quad \text{for } r = 1, 2, 3, \dots$$

Substituting these relations in (1) and aggregating the terms under the two  $\Sigma$  then gives

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2r} \frac{B_{2r} H_{2r-1}}{k+1-2r}.$$

Another proof of formula (2), independant from (1), may also be deduced from the following development given in [Ca] p. 93:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^{\mathcal{R}}(j-k) &= (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k) \\ &+ (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2} \\ &+ \int_0^1 \ln(t+1) e^{-zt} dt, \end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases} \gamma & \text{if } j = k+1 \\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.} \end{cases}$$

Rewriting the series  $\nu_k$  under the following form:

$$\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k),$$

and using the well-known relations:

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi, \quad \zeta(1-2r) = -\frac{B_{2r}}{2r},$$

as well as the combinatorial identity

$$\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{(j+1)^2} = \frac{H_{k+1}}{k+1},$$

then, a careful identification of the terms in  $\frac{z^k}{k!}$  in the previous development leads to the same formula (2) with a simpler expression of the constant  $C_k$ :

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \frac{B_{2r}}{2r(k+1-2r)}.$$

□

**Example 1.** For the first six values of  $k$ , we obtain the following relations:

$$\begin{aligned} \nu_1 &= \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1, \\ \nu_2 &= \frac{\gamma}{3} - \frac{1}{2} \ln 2\pi - 2\zeta'(-1) + \frac{2}{3}, \\ \nu_3 &= \frac{\gamma}{4} - \frac{1}{2} \ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}, \\ \nu_4 &= \frac{\gamma}{5} - \frac{1}{2} \ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}, \\ \nu_5 &= \frac{\gamma}{6} - \frac{1}{2} \ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}, \\ \nu_6 &= \frac{\gamma}{7} - \frac{1}{2} \ln 2\pi - 6\zeta'(-1) + 15\zeta'(-2) - 20\zeta'(-3) + 15\zeta'(-4) - 6\zeta'(-5) + \frac{349}{840}. \end{aligned}$$

## 2 Alternating series involving multiple zeta values

In this section, we generalize the series  $\nu_k$  to certain multiple zeta values and prove our formulae (3) and (6).

**Proposition 2.** For all natural numbers  $p \geq 0$  and integers  $k \geq -1$ , then

$$\nu_{k,p} := \sum_{j=2}^{\infty} \frac{(-1)^j}{j+k} \zeta(j, \underbrace{1, \dots, 1}_p) = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}},$$

where  $G_n^{(k)}$  are the Gregory coefficients of higher order defined by Eq. (5) or (6). In particular, for  $k \geq 0$ ,

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n}.$$

**Example 2.** For the first values of  $k$ , we obtain the following developments in series containing only positive rational terms:

$$\begin{aligned} \nu_{-1} &= 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \dots, \\ \nu_0 &= \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \dots, \\ \nu_1 &= \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25200} + \frac{731}{725760} + \dots, \\ \nu_2 &= \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100800} + \frac{5849}{10886400} + \dots, \\ \nu_3 &= \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80640} + \frac{59}{108000} + \frac{397}{1209600} + \dots, \\ \nu_4 &= \frac{1}{6} + \frac{1}{168} + \frac{5}{3024} + \frac{17}{24192} + \frac{557}{1512000} + \frac{5249}{23950080} + \dots, \\ \nu_5 &= \frac{1}{7} + \frac{1}{224} + \frac{11}{9072} + \frac{41}{80640} + \frac{439}{1663200} + \frac{311}{1995840} + \dots, \\ \nu_6 &= \frac{1}{8} + \frac{1}{288} + \frac{1}{1080} + \frac{73}{190080} + \frac{47}{237600} + \frac{2581}{22239360} + \dots, \end{aligned}$$

*Proof.* In order to prove Proposition 2 above, we use the following lemma (cf. [Xu] Eq. (2.27) and (2.28)):

**Lemma 1.** For all integers  $m \geq 1$  and  $p \geq 0$ , one has

$$\int_0^1 \frac{\ln^m(1-x) \ln^p(x)}{x} dx = (-1)^{m+p} m! p! \zeta(m+1, \underbrace{1, \dots, 1}_p).$$

Then, we can write the following equalities:

$$\begin{aligned}
\sum_{j=2}^{\infty} \frac{(-1)^j}{j+k} \zeta(j, \underbrace{1, \dots, 1}_p) &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m+k+1} \zeta(m+1, \underbrace{1, \dots, 1}_p) \\
&= \frac{(-1)^{p+1}}{p!} \sum_{m=1}^{\infty} \frac{1}{m+k+1} \int_0^1 \frac{\ln^m(1-x) \ln^p(x)}{m! x} dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{m=1}^{\infty} \frac{(-1)^m}{m+k+1} \int_0^1 \left( \sum_{n=1}^{\infty} |S_1(n, m)| \frac{x^n}{n!} \right) \frac{\ln^p(x)}{x} dx \\
&= - \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^n \frac{(-1)^m}{m+k+1} |S_1(n, m)| \frac{(-1)^p}{p!} \int_0^1 x^{n-1} \ln^p(x) dx \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n!} \sum_{m=1}^n \frac{S_1(n, m)}{m+k+1} \right) \frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}.
\end{aligned}$$

□

**Example 3.**

$$\sum_{j=2}^{\infty} \frac{(-1)^j}{j} \sum_{n=1}^{\infty} \frac{H_n}{n^j} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \sum_{n=1}^{\infty} \frac{|G_n|}{n^2} - \sum_{n=1}^{\infty} \frac{|G_n^{(0)}|}{n} + \frac{\pi^2}{6}.$$

### 3 Integral representations

In this section, we study the extension to the case of a complex parameter  $k$  and give two beautiful integral representations for the series  $\nu_k$ .

**Proposition 3.** For any  $k \in \mathbb{C}$  such that  $\operatorname{Re}(k) > -3/2$ , then

$$\nu_k = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(3/2 \pm ix + k) \cosh \pi x} dx. \quad (10)$$

For any  $k \in \mathbb{C}$  such that  $\operatorname{Re}(k) > -1/2$ , then

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix + k) \cosh \pi x} dx \quad (11)$$

By identification with formula (2), we deduce the following corollary:



**Corollary 1.** For any positive integer  $k$ , we have

$$\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix + k) \cosh \pi x} dx = \sum_{j=0}^{k-1} (-1)^{j+1} \binom{k}{j} \zeta'(-j) - C_k - \frac{1}{(k+1)^2},$$

where  $C_k$  is the rational constant defined by (9).

**Example 4.**

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(3/2 \pm ix) \cosh \pi x} dx &= \gamma, \\ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 \pm ix)}{(1/2 \pm ix) \cosh \pi x} dx &= \nu_{-1}, \\ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 \pm ix)}{(1/2 \pm ix) \cosh \pi x} dx &= -1, \\ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 + ix)}{(3/2 + ix) \cosh(\pi x)} dx &= \frac{1}{2} \ln 2\pi - \frac{5}{4}, \\ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(1/2 + ix)}{(5/2 + ix) \cosh(\pi x)} dx &= \frac{1}{2} \ln 2\pi - 2 \ln(A) - \frac{11}{18}, \end{aligned}$$

where  $A := \exp\left\{\frac{1}{12} - \zeta'(-1)\right\}$  is the Glaisher-Kinkelin constant.

*Proof.* In order to prove Proposition 3, we resort to the contour integration method. Consider the following line integral taken along a contour  $C$  consisting of the interval  $[-R, +R]$ ,  $R \in \mathbb{N}$ , on the real axis, and a semicircle of the radius  $R$  in the upper half-plane, denoted  $C_R$ ,

$$\oint_C \frac{\zeta(1/2 - iz)}{(1/2 - iz + k) \cosh \pi z} dz = \int_{-R}^{+R} \frac{\zeta(1/2 - ix)}{(1/2 - ix + k) \cosh \pi x} dx + \int_{C_R} \frac{\zeta(1/2 - iz)}{(1/2 - iz + k) \cosh \pi z} dz. \quad (12)$$

On the contour  $C_R$  the last integral may be bounded as follows:

$$\begin{aligned} \left| \int_{C_R} \frac{\zeta(1/2 - iz)}{(1/2 - iz + k) \cosh \pi z} dz \right| &= R \left| \int_0^\pi \frac{\zeta(1/2 - iRe^{i\varphi}) e^{i\varphi}}{(1/2 - iRe^{i\varphi} + k) \cosh(\pi Re^{i\varphi})} d\varphi \right| \leq \\ &\leq R \max_{\varphi \in [0, \pi]} \left| \frac{\zeta(1/2 - iRe^{i\varphi})}{1/2 - iRe^{i\varphi} + k} \right| \cdot I_R \leq \max_{\varphi \in [0, \pi]} |\zeta(1/2 - iRe^{i\varphi})| \cdot I_R \end{aligned} \quad (13)$$

where we denoted

$$I_R := \int_0^\pi \frac{d\varphi}{|\cosh(\pi R e^{i\varphi})|}, \quad R > 0,$$

for the purpose of brevity. Now, in the half-plane  $\sigma > 1$ , the absolute value of  $\zeta(\sigma + it)$  may be always bounded by a constant  $C = \zeta(\sigma)$ , which decreases and tends to 1 as  $\sigma \rightarrow \infty$ . In contrast, in the strip  $0 \leq \sigma \leq 1$  the upper bound of the function  $|\zeta(\sigma + it)|$  grows, and presently it is not known which is its exact rate of grow. However, it follows from the general theory of Dirichlet series that it cannot be faster than  $O(t^{1/4})$ . Hence, since  $\sin \varphi \geq 0$  and if  $R$  is large enough, this rough estimate gives us

$$|\zeta(1/2 - iRe^{i\varphi})| = |\zeta(1/2 + R \sin \varphi - iR \cos \varphi)| = O(\sqrt[4]{R})$$

in the interval  $\varphi \in [0, \pi]$ . On the other hand, as  $R$  tends to infinity and remains integer the integral  $I_R$  tends to zero as  $O(1/R)$ . To show this, it is sufficient to remark that

$$\frac{1}{|\cosh(\pi R e^{i\varphi})|} = \frac{\sqrt{2}}{\sqrt{\cosh(2\pi R \cos \varphi) + \cos(2\pi R \sin \varphi)}} = O(e^{-\pi R |\cos \varphi|}), \quad R \rightarrow \infty,$$

since  $0 \leq \varphi \leq \pi$  and  $R$  is integer. Thus, accounting for the symmetry of  $|\cosh(\pi R e^{i\varphi})|^{-1}$  about  $\varphi = \frac{1}{2}\pi$ , we deduce that

$$\begin{aligned} I_R &= \int_0^\pi \frac{\sqrt{2}}{\sqrt{\cosh(2\pi R \cos \varphi) + \cos(2\pi R \sin \varphi)}} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \frac{2\sqrt{2}}{\sqrt{\cosh(2\pi R \cos \varphi) + \cos(2\pi R \sin \varphi)}} d\varphi \\ &= O\left(\int_0^{\frac{\pi}{2}} e^{-\pi R \cos \varphi} d\varphi\right) = O\left(\int_0^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d\vartheta\right), \quad R \rightarrow \infty. \end{aligned} \quad (14)$$

From the well-known inequality

$$\frac{2\vartheta}{\pi} \leq \sin \vartheta \leq \vartheta, \quad \vartheta \in \left[0, \frac{1}{2}\pi\right]$$

it finally follows that

$$\frac{1 - e^{-\frac{1}{2}\pi^2 R}}{\pi R} \leq \int_0^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d\vartheta \leq \frac{1 - e^{-\pi R}}{2R}, \quad (15)$$

and since  $R$  is large, exponential terms on both sides may be neglected. Thus  $I_R = O(1/R)$  at  $R \rightarrow \infty$ .<sup>2</sup> Inserting both latter results into (13), we obtain

$$\left| \int_{C_R} \frac{\zeta(1/2 - iz)}{(1/2 - iz + k) \cosh \pi z} dz \right| = O(R^{-3/4}) \rightarrow 0 \quad \text{as } R \rightarrow \infty, R \in \mathbb{N},$$

Hence, making  $R \rightarrow \infty$ , equality (12) becomes

$$\int_{-\infty}^{+\infty} \frac{(a - ix)^{1-s}}{\cosh^2 \pi x} dx = \oint_C \frac{(a - iz)^{1-s}}{\cosh^2 \pi z} dz \quad (17)$$

where the latter integral is taken around an infinitely large semicircle in the upper half-plane. The integrand is not a holomorphic function: it has the poles of the second order at  $z = z_n \equiv i(n + \frac{1}{2})$ ,  $n \in \mathbb{N}_0$ , due to the hyperbolic secant, and a branch point at  $z = -ia$  due to the term in the numerator. If  $\text{Re}(a) > 0$ , the branch point lies outside the integration contour and we may use the Cauchy residue theorem:

$$\begin{aligned} \oint_C \frac{(a - iz)^{1-s}}{\cosh^2 \pi z} dz &= 2\pi i \sum_{n=0}^{\infty} \text{res}_{z=z_n} \frac{(a - iz)^{1-s}}{\cosh^2 \pi z} = \\ &= -\frac{2i}{\pi} \sum_{n=0}^{\infty} \frac{\partial}{\partial z} (a - iz)^{1-s} \Big|_{z=i(n+\frac{1}{2})} = \\ &= \frac{2(s-1)}{\pi} \sum_{n=0}^{\infty} (a + \frac{1}{2} + n)^{-s} = \frac{2(s-1)}{\pi} \zeta(s, a + \frac{1}{2}). \end{aligned} \quad (18)$$

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<sup>2</sup>Another way to obtain the same result is to recall that the integral (15) may be evaluated in terms of the modified Bessel function  $I_n(z)$  of the first kind and the modified Struve function  $L_n(z)$ . Using the asymptotic expansions of these special functions we obtain even a more exact result, namely

$$\int_0^{\frac{\pi}{2}} e^{-\pi R \sin \vartheta} d\vartheta = \frac{\pi}{2} \{I_0(\pi R) - L_0(\pi R)\} \sim \frac{1}{\pi R}, \quad R \rightarrow \infty, \quad (16)$$

i.e. the integral asymptotically tends to the left bound (15).

Equating (17) with the last result yields

$$\zeta\left(s, a + \frac{1}{2}\right) = \frac{\pi}{2(s-1)} \int_{-\infty}^{+\infty} \frac{(a-ix)^{1-s}}{\cosh^2 \pi x} dx, \quad \operatorname{Re}(a) > 0. \quad (19)$$

Splitting the interval of integration in two parts  $(-\infty, 0]$  and  $[0, +\infty]$  and recalling that

$$(a+ix)^s + (a-ix)^s = 2(a^2+x^2)^{\frac{s}{2}} \cos\left(s \operatorname{arctg} \frac{x}{a}\right) \quad (20)$$

the latter expression may also be written as

$$\zeta\left(s, a + \frac{1}{2}\right) = \frac{\pi}{2(s-1)} \int_0^{\infty} \frac{(a+ix)^{1-s} + (a-ix)^{1-s}}{\cosh^2 \pi x} dx \quad (21)$$

$$= \frac{\pi}{s-1} \int_0^{\infty} \frac{\cos\left[(s-1) \operatorname{arctg} \frac{x}{a}\right]}{(a^2+x^2)^{\frac{1}{2}(s-1)} \cosh^2 \pi x} dx, \quad \operatorname{Re}(a) > 0. \quad (22)$$

□

## Appendix

Let us consider the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|G_{n+r}|}{n}, \quad \text{for } r = 0, 1, 2, \dots$$

These series were the main subject of a recent article ([CY]). We have among other things established the following decomposition of  $\zeta'(-j)$  (cf. [CY], Proposition 3):

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r} (r-1)! S_2(j, r-1) \sigma_r - \frac{B_{j+1}}{j+1} \gamma - \frac{B_{j+1}}{(j+1)^2}, \quad \text{for } j = 1, 2, 3, \dots$$

where  $S_2(j, r)$  are the Stirling numbers of the second kind. Then, substituting this relation in (2) enables to write the series  $\nu_k$  as an integer linear combination of  $\sigma_1, \sigma_2, \dots, \sigma_k$  and a rational constant  $D_k$  linked to  $C_k$  (the coefficient of  $\gamma = \sigma_0$  vanishes by a well-known relation between the Bernoulli numbers<sup>3</sup>). More precisely, we obtain the following nice rewriting of formula (2):

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<sup>3</sup>The coefficient of  $\gamma$  is  $\frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j$  which is equal to zero.

$$\nu_k = D_k + \sum_{r=1}^k (-1)^r (r-1)! \left( \sum_{j=r-1}^{k-1} \binom{k}{j} S_2(j, r-1) \right) \sigma_r, \quad \text{for } k = 1, 2, 3, \dots$$

with

$$D_k = C_k - \frac{1}{2} + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2r} \frac{B_{2r}}{2r(k+1-2r)} = \frac{1}{k} - \frac{1}{2} + \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2r} \frac{B_{2r} H_{2r}}{k+1-2r}. \quad (23)$$

**Example 5.** For the first six values of  $k$ , we obtain the following relations:

$$\begin{aligned} \nu_1 &= \frac{1}{2} - \sigma_1, \\ \nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2, \\ \nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3, \\ \nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4, \\ \nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5, \\ \nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6. \end{aligned}$$

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