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# Analytic reducibility of resonant cocycles to a normal form

Claire Chavaudret <sup>\*</sup> and Laurent Stolovitch <sup>†</sup>

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## Abstract

We consider systems of quasi periodic linear differential equations associated to a ‘resonant’ frequency vector  $\omega$ , that is a vector whose coordinates are not linearly independent over  $\mathbb{Z}$ . We give sufficient conditions that ensure that a small analytic perturbation of a constant system is analytically conjugate to a ‘resonant cocycle’. We also apply our results to the non resonant case : we obtain sufficient conditions for reducibility.

## 1 Introduction

Let  $\theta = (\theta_1, \dots, \theta_d)$  be the coordinates on the torus  $\mathbb{T}^d$ . Let  $x = (x_1, \dots, x_n)$  be coordinates on  $\mathbb{R}^n$ . Let  $\omega \in \mathbb{R}^d$  and  $A_0 = (a_{i,j})_{1 \leq i, j \leq n}$  be a  $n \times n$ -matrix. Let us consider the following vector field on  $\mathbb{T}^d \times \mathbb{R}^n$ ,

$$D := \omega \frac{\partial}{\partial \theta} + A_0 x \frac{\partial}{\partial x}$$

Here, the notation  $\omega \frac{\partial}{\partial \theta}$  stands for  $\sum_{i=1}^d \omega_i \frac{\partial}{\partial \theta_i}$  and  $A_0 x \frac{\partial}{\partial x}$  stands for the vector field  $\sum_{1 \leq i, j \leq n} a_{i,j} x_j \frac{\partial}{\partial x_i}$ . We consider a linear perturbation of  $D$  of the form

$$X := D + R = \omega \frac{\partial}{\partial \theta} + A(\theta) x \frac{\partial}{\partial x} \quad (1)$$

where  $A = A_0 + a(\theta)$ ,  $a$  being an analytic matrix-valued function on  $\mathbb{T}^d$  with zero mean value:  $\hat{a}(0) = 0$ . This corresponds to the differential equation

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{x} &= A(\theta)x, \end{aligned}$$

or, in other words, it corresponds to the linear differential equations with quasi periodic coefficients

$$\frac{dx}{dt}(t) = A(\theta_0 + t\omega)x(t) \quad (2)$$

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<sup>\*</sup>Laboratoire J.-A. Dieudonné, U.M.R. 6621, Université de Nice - Sophia Antipolis, Parc Valrose 06108 Nice Cedex 02, France. courriel : [chavaudret@unice.fr](mailto:chavaudret@unice.fr)

<sup>†</sup>CNRS and Laboratoire J.-A. Dieudonné U.M.R. 6621, Université de Nice - Sophia Antipolis, Parc Valrose 06108 Nice Cedex 02, France. courriel : [stolo@unice.fr](mailto:stolo@unice.fr), This research was supported by ANR grant “ANR-10-BLAN 0102” for the project DynPDE

for some initial  $\theta_0 \in \mathbb{T}^d$ . On this vector field, there is a natural action by linear diffeomorphisms over the torus: if  $Z : \mathbb{T}^d \rightarrow gl_n(\mathbb{C})$  is a matrix-valued function with zero mean value, we can consider the (linear) change of variables  $Z = (\text{Id}_n + X(\theta))Y$  over the torus. It conjugates (2) to

$$\frac{dx}{dt}(t) = A'(\theta_0 + t\omega)x(t). \quad (3)$$

where  $A'$  is the matrix valued function satisfying to

$$DX(\theta).\omega = A(\theta)(\text{Id}_n + X(\theta)) - (\text{Id}_n + X(\theta))A'(\theta). \quad (4)$$

This can readily be seen by differentiating with respect to  $t$ , the quantity

$$Z(\theta_0 + t\omega) = (\text{Id}_n + X(\theta_0 + t\omega))Y(\theta_0 + t\omega)$$

We will denote  $DX(\theta).\omega$  by  $\partial_\omega X(\theta)$ .

We are interested in the problem of analytic classification with respect to such a group action. A case of particular interest is the one for which the cocycle  $A$  can be transformed to a **constant** matrix. In that situation, we say that  $A$  is **reducible**. The regularity class of the transformations used is important. This reducibility phenomenon is well known for cocycles over the circle (Floquet theory, see [Arn80][chap. 26]). When the base is a higher dimensional torus, the situation is much more complicated. The frequencies  $\omega = (\omega_1, \dots, \omega_d)$  are usually assumed to be linearly independent over  $\mathbb{Z}$ . In that case, they are also assumed to satisfy a ‘Diophantine’ condition.

Most known results consider the case where the ‘fiber’ has dimension  $n = 2$ . In that case, the rotation number [JM82] of the cocycle can be defined. The rotation number is a very useful quantity since its arithmetical properties determine whether the cocycle can be reduced or not: in the analytic category, [Eli92] proved reducibility for Schrödinger cocycles with a large energy or a small potential, under a Siegel-type Diophantine condition on the frequency vector and on the rotation number; a similar result was obtained in [CM12] under weaker arithmetical conditions, namely, Brjuno-Rüssmann conditions, on the frequencies and on the rotation number. Non-perturbative versions of Eliasson’s results were also obtained: the article [AK06] gives a description of Schrödinger cocycles with a recurrent Diophantine frequency vector for almost every energy.

In a higher dimensional fiber, reducibility results and reducibility criteria were obtained in [Kri99a, Kri99b] when the fiber is in a compact Lie group; in particular, reducibility in full measure for a one-parameter family of cocycles under non-degeneracy conditions and the equivalence between reducibility and compactness of the iterates. The authors of [HY08] proved a similar result in the non compact case. Almost reducibility of cocycles in any dimension, that is, the possibility of conjugating them to a cycle arbitrarily close to a constant, was also proved in a perturbative framework in [Eli01] and improved in [Cha13]. However, positive reducibility results still seem to depend on the existence of a rotation number, or on strong spectral assumptions as in [MS65].

The previously mentioned articles all require Diophantine or Brjuno conditions on the frequency vector. In [ZW12] and in [AFK11], the case of a liouvillean frequency vector was considered. However, the results are only stated for a one or two-dimensional frequency vector.

In this article, we shall consider analytic cocycles over a torus  $\mathbb{T}^d$  with a **fiber of any dimension**  $n$ . We consider them as ‘small’ perturbations of constant cocycles. We consider the general case where

the frequencies might be linearly dependent over  $\mathbb{Z}$ , that is there are **resonances**. We cannot expect such a general cocycle to be reducible over the torus. We shall give sufficient conditions that ensure that the cocycle is analytically conjugate to an analytic **resonant cocycle**, that is a cocycle in which Fourier modes  $e^{i\langle k, \theta \rangle}$ , depend only on the resonances relations of the frequencies, that is  $\langle k, \omega \rangle = 0$ . In particular, a resonant cocycle is constant along resonant trajectories of the torus.

Our first result concerns perturbations of a diagonal matrix with separated spectrum, and can be viewed as a generalization of [MS65] and [Adr62].

**Theorem 1.1.** *Let  $A_0$  be a diagonal matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$  having distinct real parts and let  $C = \min_{j \neq k} \operatorname{Re} \lambda_j - \operatorname{Re} \lambda_k > 0$ . Assume that  $\omega$  is very weak with exponent  $R > 0$  (see definition 2.8). Let  $r > R$  and let  $a$  be a  $C_r^\omega$  matrix valued function with zero mean value. There exists  $\varepsilon_0(r, n, d, \omega, C)$  such that if  $|a|_r \leq \varepsilon_0$ , then the system whose coefficients are  $A_0 + a$  can be analytically conjugated to a system whose coefficient matrix is resonant and diagonal.*

**Corollary 1.2.** *Under assumptions of the previous theorem, equation (1) is analytically reducible to a constant cocycle on each level set  $\cap_i \{\theta \mid \langle m_i, \theta \rangle = \text{constant}_i\}$ .*

**Corollary 1.3.** *Under assumptions of the previous theorem and if there is no resonances relations (i.e.  $\mathcal{R} = \{0\}$ ), then the cocycle  $A$  is analytically reducible.*

This result is due to L. Adrianova [Adr62] and to Mitropolskii-Samoilenko ([MS65]) when the frequencies satisfy a Siegel's type small divisors condition. In our result, we only require a much weaker condition (even much weaker than Brjuno-Rüssmann condition).

The second main result considers triangular perturbations of a constant system.

**Theorem 1.4.** *Let  $S$  be a diagonal matrix which satisfies a second Melnikov condition away from the resonances (see definition 2.9). Let  $F$  be a  $C_r^\omega$  matrix valued function, with upper triangular and nilpotent values. There exists  $\varepsilon_0(n, d, \omega, S, r)$  such that if  $|F - \hat{F}(0)|_r \leq \varepsilon_0$ , then the system whose coefficients are given by  $S + F$  can be analytically conjugated to a system whose coefficients matrix is resonant, upper triangular and commuting with  $S$ .*

As a consequence, we obtain a decomposition into analytic invariant subbundles of the initial cocycles defined by the unperturbed constant cocycle.

Next we will consider strongly commuting perturbations, i.e perturbations which commute with the constant part and also such that their various Fourier modes all commute with each other (see definition 5.1).

**Theorem 1.5.** *Assume that the vector  $\omega$  is very weak with exponent  $R > 0$ . Let  $A_0 \in gl(n, \mathbb{C})$ . Let  $r > R$  and let  $F \in C_r^\omega$  be matrix valued and strongly commuting, and such that all Fourier modes of  $F$  commute with  $A_0$ . There exists  $C(\omega, A_0, r, n, d)$  such that if  $|F|_r \leq C$ , then the system  $A_0 + F$  is reducible to a resonant system  $A_\infty(\theta)$  which commutes with  $A_0$ .*

Notice that, under this strong algebraic condition on the perturbation, no second Melnikov condition is needed since most small divisors will be avoided. In the last section, we shall consider similar statements under assumptions that the cocycles belongs to a Lie sub algebra of matrices.

We shall also apply our results to a non-resonant situation to obtain sufficient conditions that ensure analytic reducibility. Our proofs are based on a Newton-KAM scheme whose convergence is due to the control of the small divisors of the initial unperturbed constant cocycle.

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## 2 Notations and assumptions

**Definition 2.1.** *The vector  $\omega$  is said to be **resonant** if there exists  $m_1, \dots, m_s \in \mathbb{Z}^d$  linearly independent over  $\mathbb{Q}$  such that  $\langle m_i, \omega \rangle = 0, i = 1, \dots, s$ , and if  $\langle k, \omega \rangle = \omega_i$  then  $k = E_i + \sum_{j=1}^s l_j m_j$  where the  $l_j$ 's are integers and  $E_i$  is the  $i$ th-unit vector of  $\mathbb{Z}^d$ . We shall denote by  $\mathcal{R}$  the  $\mathbb{Z}$ -module generated by the  $m_i$ 's. We define an equivalence relation  $\sim$  on  $\mathbb{Z}^d$  by*

$$k_1 \sim k_2 \Leftrightarrow \exists l \in \mathbb{Z}^s, k_1 = k_2 + l \cdot m \Leftrightarrow k_1 - k_2 \in \mathcal{R}$$

*The equivalence class of an integer vector  $v$  will be denoted by  $\langle v \rangle$ .*

**Remark 2.2.** *A sum is defined on equivalence classes as follows:  $\langle u \rangle + \langle v \rangle = \langle u + v \rangle$ ; this is well defined since  $u \sim u', v \sim v' \Rightarrow u + v \sim u' + v'$ .*

**Definition 2.3.** *A function  $f$  defined on the torus is said to be **resonant** if its Fourier series has only modes which are proportional to the  $m_j$ 's, i.e.  $f = \sum_{l \in \mathbb{Z}^s} f_l e^{i \langle l \cdot m, \theta \rangle}$ , where  $l \cdot m$  stands for  $l_1 m_1 + \dots + l_s m_s$ . In that case, we shall write  $f_{res}$  to notify this fact.*

**Definition 2.4.** *Let  $\mathcal{H}_{\langle v \rangle}$  be the subspace of functions  $f$  defined on the torus whose elements only have Fourier modes in  $\langle v \rangle$ .*

**Remark 2.5.** *For all function  $f$  continuous on the torus, there exists a unique decomposition  $f = f_{res} + f_{nr}$  where  $f_{res}$  is resonant and  $f_{nr}$  does not have any harmonics in the lattice  $\mathcal{R}$  generated by the  $m_j$ 's. More generally, there is a natural decomposition of the space of continuous functions on the torus into subspaces  $\mathcal{H}_{\langle v \rangle}: f = \sum_{\langle v \rangle \in \mathbb{Z}^d / \sim} f_{\langle v \rangle}$ . With this notation,  $f_{res}$  coincides with  $f_{\langle 0 \rangle}$ . Note that  $\mathcal{H}_{\langle v \rangle} \cdot \mathcal{H}_{\langle v' \rangle} \subset \mathcal{H}_{\langle v+v' \rangle}$ .*

**Definition 2.6.** *For a function  $f$  defined on  $\mathbb{T}^d$  (be it matrix-valued or scalar-valued), if the Fourier modes of  $f$  belong to a finite number of equivalence classes, then we shall denote by  $\deg F$  the **degree** of  $f$ , i.e the smallest integer such that all equivalence classes of Fourier modes of  $f$  have a representative  $v = (v_1, \dots, v_d)$  with length  $|v_1| + \dots + |v_d| \leq \deg F$ .*

When necessary, for a matrix-valued function  $F$ , we will denote by  $F_{diag}$  its diagonal part.

We will consider the Banach space  $C_r^\omega$  of (matrix valued) functions which are analytic on a neighborhood  $\{|\operatorname{Im} \theta| \leq r\}$  of the torus such that

$$|f|_r := \sum_{\langle v \rangle \in \mathbb{Z}^d / \sim} |f_{\langle v \rangle}|_r < +\infty, \quad |f_{\langle v \rangle}|_r := \sup_{|\operatorname{Im} \theta| \leq r} \|f_{\langle v \rangle}(\theta)\|$$

where  $\|\cdot\|$  denotes the operator norm on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . We notice that this norm is sub-multiplicative, i.e.  $|fg|_r \leq |f|_r |g|_r$ .

**Assumption 2.7.** *The function  $A$  in equation (1) is analytic in a neighborhood of the torus (in fact, in a complex strip around the real  $d$ -dimensional plane) so there exists  $r > 0$  such that  $A \in C_r^\omega$ .*

**Definition 2.8.** *We shall say that the frequency vector  $\omega$  is **very weak** if there exist  $R > 0, K > 0$  such that for all  $k \in \mathbb{Z}^d$  with  $\langle k, \omega \rangle \neq 0$ , then*

$$|\langle k, \omega \rangle| \geq K e^{-R|k|}. \quad (5)$$

*If this holds, the number  $R$  is simply called the exponent of  $\omega$ .*

This kind of very weak condition also appeared in different context in [Sto97].

**Definition 2.9.** *We shall say that  $A_0$  is a **Melnikov matrix** or satisfies the second Melnikov condition away from the resonances if there exist a constant  $\kappa'$  and a strictly increasing differentiable function  $g' : [1, +\infty[ \rightarrow \mathbb{R}^{*+}$  satisfying*

$$\int_1^\infty \frac{\ln(g'(t))}{t^2} dt < +\infty$$

*such that for all pair  $(\alpha, \beta)$  of eigenvalues of  $A_0$ , and all  $m' \in \mathbb{Z}^d$  such that  $i\langle m', \omega \rangle - \alpha + \beta \neq 0$ , then*

$$|\alpha - \beta - i\langle m', \omega \rangle| \geq \frac{\kappa'}{g'(|m'|)}. \quad (6)$$

*We shall also say that  $A_0$  is Melnikov up to order  $N \in \mathbb{N}$  if Equation (6) holds for all  $m'$  with  $|m'| \leq N$ .*

**Remark 2.10.** *If  $A_0$  is a Melnikov matrix, then in particular  $\omega$  satisfies a Brjuno-Rüssmann arithmetical condition away from the resonances, which is weaker than Siegel's diophantine condition, and was used for instance in [CM12].*

### 3 Analytic conjugation to a normal form: the separated diagonal case

Let us make an extra assumption:

**Assumption 3.1.**  *$A_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$  is diagonal with eigenvalues having distinct real parts.*

The main result of this section is the following :

**Theorem 3.2.** *Assume that  $\omega$  is very weak with exponent  $R > 0$ . Let  $r > R$  and let  $A = A_0 + a$  be a  $C_r^\omega$ -analytic cocycle (1) where  $a$  is a zero-mean valued matrix function. There exists  $\varepsilon_0$  depending only on  $A_0, n$  such that if  $|a|_r \leq \varepsilon_0$ , then there exists a convergent transformation conjugating (1) to a normal form in a neighborhood of the torus:*

$$NF = \omega \frac{\partial}{\partial \theta} + D_{res}(\theta)x \frac{\partial}{\partial x}.$$

where  $D_{res}$  is a resonant diagonal matrix-valued function.

We shall prove Theorem 3.2 in two steps: first, the system can be diagonalized without any arithmetical assumption on  $\omega$ ; second, such a diagonal system can be conjugated to a resonant one under the very mild arithmetical assumption (5) on  $\omega$  (it is much weaker than the Brjuno-Rüssmann condition contained in the definition 2.9). The precise statements are as follows:

**Proposition 3.1.** *Suppose that  $A_0$  has eigenvalues with distinct real parts and that  $a$  is a matrix-valued function, analytic on the torus, with zero mean value. Then there exists  $\varepsilon_0$  depending only on  $A_0, n$  such that if  $|a|_r \leq \varepsilon_0$ , then there exists an analytic transformation conjugating the system whose coefficient matrix is  $A_0 + a$  to a diagonal system.*

Proposition 3.1 will be proved in the following sections. The second step is much easier:

**Proposition 3.2.** *Let  $D$  be a  $C_r^\omega$ -analytic function on the torus whose values are diagonal matrices; if there exist  $r' > 0, C' > 0$  such that for all  $k \in \mathbb{Z}^d$  with  $\langle k, \omega \rangle \neq 0$ ,*

$$|\langle k, \omega \rangle| \geq C' e^{-|k|(r-r')}$$

*then the system with coefficient matrix  $D$  can be  $C_{r'}^\omega$ -conjugated to a resonant diagonal system.*

*Proof.* Consider the system with analytic coefficients given by  $D$ , that is,

$$\begin{cases} \dot{x}_1(\theta, t) = D_{1,1}(\theta + t\omega)x_1(\theta, t) \\ \vdots \\ \dot{x}_n(\theta, t) = D_{n,n}(\theta + t\omega)x_n(\theta, t) \end{cases} \quad (7)$$

Each line can be solved separately and immediately by

$$x_j(\theta, t) = \exp\left(\int_0^t D_{j,j}(\theta + s\omega)ds\right) x_j(\theta, 0)$$

By setting  $D_{j,j}(\theta) = D_{j,j,res}(\theta) + D_{j,j,nr}(\theta)$ , one has

$$x_j(\theta, t) = \exp(tD_{j,j,res}(\theta)) \exp\left(\int_0^t D_{j,j,nr}(\theta + s\omega)ds\right) x_j(\theta, 0).$$

Indeed, we have

$$D_{j,j,res}(\theta) = \sum_{l \in \mathbb{Z}^s} D_{j,j,l} e^{i\langle l \cdot m, \theta \rangle},$$

where  $l \cdot m$  stands for  $l_1 m_1 + \dots + l_s m_s$ . Therefore,  $D_{j,j,res}(\theta + s\omega) = D_{j,j,res}(\theta)$ . One can explicit

$$\int_0^t D_{j,j,nr}(\theta + s\omega) = \sum_{k \in \mathbb{Z}^d} \widehat{D}_{j,j,nr}(k) \frac{e^{i\langle k, \theta + t\omega \rangle} - e^{i\langle k, \theta \rangle}}{i\langle k, \omega \rangle}$$

Letting now  $D_{res}(\theta) = \text{diag}(D_{j,j,res}(\theta))$  and letting  $Z$  be the diagonal-valued function given by

$$Z_{j,j}(\theta) = \exp \left( \sum_{k \in \mathbb{Z}^d, \langle k, \omega \rangle \neq 0} \widehat{D}_{j,j}(k) \frac{e^{i\langle k, \theta \rangle}}{i\langle k, \omega \rangle} \right)$$

gives the desired conjugation

$$X(\theta, t) = Z(\theta + t\omega) e^{tD_{res}(\theta)} Z(\theta)^{-1}$$

The question therefore reduces to finding an arithmetical condition on  $\omega$  under which  $Z$  is analytic. Since  $D$  is assumed to be analytic, then for every  $k$ ,

$$|\widehat{D}_{j,j}(k)| \leq |D_{j,j}|_r e^{-|k|r}$$

thus  $Z$  is analytic whenever there exist  $r' > 0$  and  $C > 0$  such that for all  $k \in \mathbb{Z}^d, \langle k, \omega \rangle \neq 0$ ,

$$e^{-|k|r} |\langle k, \omega \rangle|^{-1} \leq C e^{-|k|r'}$$

that is to say,

$$|\langle k, \omega \rangle| \geq C' e^{-|k|(r-r')}.$$

□

### 3.1 Cohomological equation and iteration process

The proof of Proposition 3.1 is based on the following:

**Proposition 3.3.** *Let  $A : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  be a diagonal perturbation of  $A_0$ . Assume  $\inf_{|\text{Im } \theta| \leq r} |Re A(\theta)_{j,j} - Re A(\theta)_{k,k}| \geq \delta > 0$  for all  $v \in \mathbb{Z}^d$  and all  $j \neq k$ . Let  $F : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  be a  $C_r^\omega$ -analytic matrix-valued function. There exists a constant  $C_n$  only depending on  $n$  such that if*

$$|A - A_0|_r < \frac{\delta}{8C_n}$$

then there exists a solution  $X : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  of the equation

$$\partial_\omega X(\theta) = [A(\theta), X(\theta)] + F(\theta) - F_{diag}(\theta) \tag{8}$$

where  $\partial_\omega$  stands for the derivative in the direction  $\omega$ .

Moreover,  $X$  is analytic and belongs to  $C_r^\omega$ , with the estimate

$$|X|_r \leq \frac{4C_n |F|_r}{3\delta - 8C_n |A - A_0|_r} \leq \frac{2C_n}{\delta} |F|_r.$$



*Proof.* Equation (8) is equivalent to

$$(\mathcal{D} + \mathcal{R})(X)(\theta) = F(\theta) - F_{diag}(\theta) \quad (9)$$

where  $\mathcal{D} : X \mapsto \partial_\omega X - [A_{res}, X]$  and  $\mathcal{R} : X \mapsto -[A_{nr}, X]$ . One has the bound  $|\mathcal{R}X|_r \leq 2|A - A_0|_r |X|_r$ . Concerning  $\mathcal{D}$ , an equation of the form  $\mathcal{D}X = G - G_{diag}$  is equivalent to its decomposition along the  $\mathcal{H}_{\langle v \rangle}$ 's: for all  $\langle v \rangle \in \mathbb{Z}^d / \sim$  such that  $\langle v, \omega \rangle \neq 0$ ,

$$i\langle v, \omega \rangle X_{\langle v \rangle}(\theta) = [A_{res}(\theta), X_{\langle v \rangle}(\theta)] + G_{\langle v \rangle}(\theta) - G_{\langle v \rangle, diag}(\theta). \quad (10)$$

By decomposing by matrix coefficients, we obtain, if  $j = k$ , that  $(G - G_{diag})_{\langle v \rangle, j, j} = 0$  and we can set  $X_{\langle v \rangle}(\theta)_{j, j} := 0$ . For  $j \neq k$ , and all  $v, \theta$ ,

$$X_{\langle v \rangle}(\theta)_{j, k} = \frac{G_{\langle v \rangle}(\theta)_{j, k}}{i\langle v, \omega \rangle - A(\theta)_{j, j, res} + A(\theta)_{k, k, res}},$$

If  $\langle v \rangle \in \mathbb{Z}^d / \sim$  is such that  $\langle v, \omega \rangle = 0$ , then for  $j = k$ , we select  $X_{\langle v \rangle}(\theta)_{j, j} := 0$ . For  $j \neq k$ , we have

$$X_{\langle v \rangle}(\theta)_{j, k} = \frac{G_{\langle v \rangle}(\theta)_{j, k}}{-A(\theta)_{j, j, res} + A(\theta)_{k, k, res}}.$$

Since for all  $j \neq k$  and for all  $v \in \mathbb{Z}^d$ , we have

$$\inf_{|\text{Im } \theta| \leq r} |A(\theta)_{j, j, res} - A(\theta)_{k, k, res} - i\langle v, \omega \rangle| \geq \inf_{|\text{Im } \theta| \leq r} |\text{Re}A(\theta)_{j, j, res} - \text{Re}A(\theta)_{k, k, res}|,$$

since we have

$$A(\theta)_{j, j, res} = A(\theta)_{j, j} + (A(\theta)_{j, j, res} - A(\theta)_{j, j})$$

and

$$\sup_{|\text{Im } \theta| \leq r} |A(\theta)_{j, j, res} - A(\theta)_{j, j}| \leq |A(\theta)_{j, j, res} - A(\theta)_{j, j}|_r \leq C_n |A_0 - A(\theta)|_r,$$

then we have

$$\inf_{|\text{Im } \theta| \leq r} |A(\theta)_{j, j, res} - A(\theta)_{k, k, res} - i\langle v, \omega \rangle| \geq \delta - \frac{C_n \delta}{4C_n} = \delta \frac{3}{4}$$

as well as

$$|X_{\langle v \rangle}(\theta)_{j, k}|_r = \sup_{|\text{Im } \theta| \leq r} |X_{\langle v \rangle}(\theta)_{j, k}| \leq \frac{4}{3} \sup_{|\text{Im } \theta| \leq r} \frac{|G_{\langle v \rangle, j, k}(\theta)|}{\delta} \leq \frac{4}{3} C_n \frac{|G_{\langle v \rangle}|_r}{\delta}.$$

Therefore, the operator  $\mathcal{D}$  is invertible and one has the bound  $|\mathcal{D}^{-1}|_{C_r^\omega \rightarrow C_r^\omega} \leq \frac{4C_n}{3\delta}$ . Therefore, if  $|A - A_0|_r < \frac{\delta}{8C_n}$ , then

$$\begin{aligned}
|(\mathcal{D} + \mathcal{R})^{-1}|_r &\leq |\mathcal{D}^{-1}|_r |(Id + \mathcal{D}^{-1}\mathcal{R})^{-1}|_r \\
&\leq \frac{4C_n}{3\delta} \sum_{j \geq 0} \left( \frac{4C_n}{3\delta} 2|A - A_0|_r \right)^j \\
&\leq \frac{4C_n}{3\delta - 8C_n|A - A_0|_r} \leq \frac{2C_n}{\delta},
\end{aligned}$$

thus the equation (8) has a solution  $X$  satisfying  $|X|_r \leq \frac{2C_n}{\delta}|F|_r$ .  $\square$

$\square$

**Proposition 3.4.** *[Induction argument] Let  $A : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  be a diagonal analytic perturbation of  $A_0$ . Let  $\delta$  and  $\varepsilon < \frac{\delta}{4C_n}$  be positive numbers. Let  $F : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  belong to  $C_r^\omega$  with*

$$|F|_r \leq \varepsilon \quad (11)$$

Let  $C_n$  be the constant defined in the assumptions of Proposition 3.3. Assume that  $A$  satisfies to

$$\inf_{|\operatorname{Im} \theta| \leq r} |\operatorname{Re} A(\theta)_{j,j} - \operatorname{Re} A(\theta)_{k,k}| \geq \delta \quad (12)$$

for all  $v \in \mathbb{Z}^d$ , and to

$$|A - A_0|_r \leq \frac{\varepsilon}{2} \leq \frac{\delta}{8C_n}. \quad (13)$$

There exists  $A' : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  a diagonal  $C_r^\omega$ -analytic perturbation of  $A_0$ , there exists  $F' : \mathbb{T}^d \rightarrow gl(n, \mathbb{C})$  which belongs to  $C_r^\omega$ , and there exists  $Z : \mathbb{T}^d \rightarrow GL(n, \mathbb{C})$  which belongs to  $C_r^\omega$  such that for all  $\theta \in \mathbb{T}^d$ ,

$$\partial_\omega Z(\theta) = (A(\theta) + F(\theta))Z(\theta) - Z(\theta)(A'(\theta) + F'(\theta))$$

and

$$|F'|_r \leq \frac{8C_n}{\delta} |F|_r^2$$

Moreover

$$|Z - I|_r \leq \frac{2C_n}{\delta} |F|_r \quad (14)$$

and  $A'$  satisfies

$$\inf_{|\operatorname{Im} \theta| \leq r} |\operatorname{Re} A'(\theta)_{j,j} - \operatorname{Re} A'(\theta)_{k,k}| \geq \delta - 2C_n \varepsilon \quad (15)$$

for all  $v \in \mathbb{Z}^d$ , as well as

$$|A' - A_0|_r \leq \frac{\varepsilon}{2} + \varepsilon \quad (16)$$

*Proof.* Let  $X$  be the solution of  $\partial_\omega X = [A, X] + F - F_{diag}$  given by Proposition 3.3 and let us set  $Z := I + X$ . Then, according to (4), we have

$$\partial_\omega Z(\theta) = (A(\theta) + F(\theta))Z(\theta) - Z(\theta)(A'(\theta) + F'(\theta)),$$

where

$$\begin{aligned} A' &:= A + F_{diag} \\ F' &:= (I + X)^{-1}FX + \sum_{l \geq 1} (-X)^l F_{diag}. \end{aligned}$$

By definition,  $A'$  is diagonal and the estimate (16) comes directly from (13). According to (12), applying Proposition 3.3, we obtain

$$|X|_r \leq \frac{2C_n}{\delta} |F|_r \leq \frac{1}{2}$$

where  $C_n$  only depends on  $n$  (which also gives the estimate (14)).

Thus we have

$$\begin{aligned} |F'|_r &\leq |(I + X)^{-1}(FX - XF_{diag})|_r \\ &\leq \left( \sum_{l \geq 0} |X|_r^l \right) \frac{4C_n}{\delta} |F|_r^2 \end{aligned} \tag{17}$$

$$\leq \frac{8C_n}{\delta} |F|_r^2. \tag{18}$$

Moreover, for any scalar  $f \in C_r^\omega$ , we have  $\sup_{|\operatorname{Im} \theta| \leq r} |f(\theta)| \leq |f|_r$ . As a consequence, we have, for  $|\operatorname{Im} \theta| \leq r$  and for all  $j \neq k$ ,

$$\begin{aligned} |\operatorname{Re} A'(\theta)_{j,j} - \operatorname{Re} A'(\theta)_{k,k}| &\geq (|\operatorname{Re} A(\theta)_{j,j} - \operatorname{Re} A(\theta)_{k,k}|) - (|\operatorname{Re}(F(\theta))_{j,j}| + |\operatorname{Re}(F(\theta))_{k,k}|) \\ &\geq (|\operatorname{Re} A(\theta)_{j,j} - \operatorname{Re} A(\theta)_{k,k}|) - (|(F)_{j,j}|_r + |(F)_{k,k}|_r) \end{aligned}$$

Hence, we have

$$\inf_{|\operatorname{Im} \theta| \leq r} |\operatorname{Re} A'(\theta)_{j,j} - \operatorname{Re} A'(\theta)_{k,k}| \geq \delta - 2C_n \varepsilon$$

□

### 3.2 Proof of Proposition 3.1

We prove this by iterating Proposition 3.4 and constructing a sequence of changes of variables, all  $C_r^\omega$ -analytic, conjugating the system  $A_0 + a$  to something which is arbitrarily close to the system given in the statement.

Let  $C_n$  be the constant defined in Proposition 3.3. Let  $\delta = \delta_0$  and  $\varepsilon = \varepsilon_0 \leq 1/2$  be such that

$$\varepsilon_0 \leq \varepsilon_0^{1/2} \leq \frac{\delta_0}{8C_n}$$

and

$$\min_{j \neq k} |ReA_{0,j,j} - ReA_{0,k,k}| \geq \delta_0.$$

Let us set  $\varepsilon_k := \varepsilon_0^{(3/2)^k}$  as well as

$$\delta_k := \delta_{k-1} - 2C_n \varepsilon_{k-1} = \delta_0 - 2C_n \sum_{j=0}^{k-1} \varepsilon_j, \quad k \geq 1.$$

Notice that, if  $\varepsilon$  is small enough, we have  $\delta_k > 16C_n \varepsilon_0$ , for all  $k \geq 0$ . Indeed, for  $k \geq 1$  we have  $\varepsilon_0^{(3/2)^k} \leq \varepsilon_0^k$  so that

$$\sum_{j \geq 0} \varepsilon_j \leq \varepsilon_0 + \frac{\varepsilon_0}{1 - \varepsilon_0} \leq 3\varepsilon_0.$$

Therefore, we have, if  $\varepsilon_0$  is small enough

$$\delta_k \geq 2C_n(4\varepsilon_0^{1/2} - 3\varepsilon_0) \geq 16C_n \varepsilon_0.$$

Assumptions 2.7 and 3.1 make it possible to apply Proposition 3.4 with  $A = A_0$  and  $F = a$ : if  $|a|_r \leq \varepsilon_0$ , we obtain a conjugation  $Z_1$  to a new vector field  $D_1 + R_1$  with  $D_1 = \omega \frac{\partial}{\partial \theta} + A_1(\theta)x \frac{\partial}{\partial x}$ , where  $A_1(\theta)$  is diagonal with  $\delta_1 = \delta_0 - 2C_n \varepsilon_0$ -separated spectrum, and  $R_1$  is  $C_r^\omega$ -analytic on the torus with

$$|R_1|_r \leq 8 \frac{C_n}{\delta_0} |a|_r^2 \leq \varepsilon_0^{\frac{3}{2}} = \varepsilon_1$$

and

$$|A_1 - A_0|_r \leq \frac{3}{2} \varepsilon_0 \leq \frac{\delta_1}{2C_n}$$

The change of variable  $Z_1$  is itself  $C_r^\omega$ -analytic on the torus, and  $\frac{\varepsilon_0^{\frac{1}{2}}}{4}$ -close to the identity since

$$\frac{2C_n}{\delta_0} |a|_r \leq \frac{\varepsilon_0^{\frac{1}{2}}}{4}.$$

Now suppose that we have a change of variables  $Z_k$  which conjugates the vector field  $D + R$  given by (1) to  $D_k + R_k$  where  $D_k(\theta) = \omega \frac{\partial}{\partial \theta} + A_k(\theta)x \frac{\partial}{\partial x}$  and  $A_k$  is diagonal with  $\delta_k$ -separated spectrum, with the estimates

$$|R_k|_r \leq \varepsilon_k,$$

$$|A_k - A_0|_r \leq \sum_{j=0}^{k-1} \varepsilon_j < 2\varepsilon_0 \leq \frac{\delta_k}{8C_n}$$

and

$$|\tilde{Z}_k - Id|_r \leq 2C \sum_{j=0}^{k-1} \varepsilon_j^{1/2}$$

We remark, if  $\varepsilon_0$  is small enough, then, for all  $k \geq 1$ , we have  $\varepsilon_k^{1/2} < \frac{\delta_k}{8C_n}$ . Indeed, we have

$$\frac{\delta_k}{8C_n} \geq 2C_n \varepsilon_0 \geq \varepsilon_0^{(1/2)(3/2)^k}.$$

Thus, one can apply again Proposition 3.4 to conjugate  $D_k + R_k$  by some change of variable  $Z_k$  such that

$$|Z_k - Id|_r \leq \frac{2C_n}{\delta_k} \varepsilon_k \leq \varepsilon_k^{\frac{1}{2}}$$

to a vector field  $D_{k+1} + R_{k+1}$  with analogous properties:

$$D_{k+1}(\theta) = \omega \frac{\partial}{\partial \theta} + A_{k+1}(\theta) x \frac{\partial}{\partial x}$$

where  $A_{k+1}$  is diagonal, with  $\delta_{k+1}$ -separated spectrum,

$$|R_{k+1}|_r \leq \frac{8C_n}{\delta_k} \varepsilon_k^2 \leq \varepsilon_{k+1},$$

$$|A_{k+1} - A_0|_r \leq |A_k - A_0|_r + \varepsilon_k \leq \sum_{j=0}^k \varepsilon_j \leq \frac{\delta_{k+1}}{8C_n}$$

so that  $D + R$  is conjugated to  $D_{k+1} + R_{k+1}$  by  $\tilde{Z}_{k+1} = Z_k \tilde{Z}_k$ .

In  $C_r^\omega$ ,  $R_k$  tends to 0 and  $\{D_k\}_k$  is a Cauchy sequence thus it converges to a limit  $D_\infty$  which is diagonal and analytic. Since  $\tilde{Z}_k$  is  $C_r^\omega$ -analytic for all  $k$  and defines a Cauchy sequence, then it has an analytic limit  $Z_\infty$  conjugating  $D + R$  to  $D_\infty$ . □

## 4 Analytic reduction of an upper triangular perturbation

In this section, we assume that  $A_0 = S + M$ , where  $S = \text{diag}(\lambda_j)$  is diagonal and  $M$  is upper triangular and commutes with  $S$ . This assumption is not restrictive since  $A_0$  can be conjugated from the start to its Jordan normal form. In Section 6, we will see how to preserve real structures while still using the Jordan normal form.

The main result of this section is :

**Theorem 4.1.** *Let  $A_0 = S + M \in gl(n, \mathbb{C})$  where  $S = \text{diag}(\lambda_j)$  is diagonal,  $M$  is nilpotent and upper triangular and  $[M, S] = 0$ . Let  $F \in C_r^\omega$  be upper triangular valued. Suppose  $S$  is Melnikov (see definition 2.9). There exists  $\varepsilon_0(n, d, \kappa', g', A_0, r)$  such that if  $|F|_r \leq \varepsilon_0$ , then there exists  $r_\infty > 0$  and  $Z_\infty \in C_{r_\infty}^\omega$  and an upper triangular, resonant  $R_\infty \in C_{r_\infty}^\omega$  such that  $\partial_\omega Z_\infty = (A_0 + F)Z_\infty - Z_\infty R_\infty$ .*

**Remark 4.2.** *If  $A_0$  is not in Jordan normal form, the assumption on  $F$  is that it is conjugate to an upper triangular valued function by the conjugation which takes  $A_0$  to its Jordan normal form.*

## 4.1 Cohomological equation and iteration process

**Definition 4.3.** Let  $l \leq n$  and let  $\mathcal{F}_l$  be the set of  $n \times n$ -matrices  $M = (m_{i,j})_{1 \leq i,j \leq n}$  such that  $m_{i,j} = 0$  if  $j < i + l$ . We have  $\mathcal{F}_n = \{0\}$ .

Let us start with an elementary lemma :

**Lemma 4.4.** We have  $[\mathcal{F}_l, \mathcal{F}_p] \subset \mathcal{F}_{p+l}$ .

*Proof.* Let  $M \in \mathcal{F}_l$  and  $N \in \mathcal{F}_p$ . Then, we have

$$[N, M]_{i,j} := \sum_{k=1}^n n_{i,k} m_{k,j} - \sum_{k=1}^n m_{i,k} n_{k,j} = \sum_{k=i+l}^n n_{i,k} m_{k,j} - \sum_{k=i+p}^n m_{i,k} n_{k,j}.$$

In the first sum,  $m_{k,j} = 0$  if  $j < k + p$  so the sum is zero if  $j < (i+l) + p \leq k + p$ . By the same argument, the second sum is zero for  $j < (i+p) + l \leq k + l$ .  $\square$

We will now give a lemma (appearing in [Eli01] for the non resonant case) which shows that, up to a simple analytic change of variables, one can assume that  $S = \text{diag}(\lambda_j)$  satisfies

$$i\langle v, \omega \rangle - \lambda_k + \lambda_l = 0 \Rightarrow i\langle v, \omega \rangle = 0$$

**Lemma 4.5.** Let  $S = \text{diag}(\lambda_j)$  be a diagonal matrix and  $N$  commuting with  $S$ . There exists an analytic change of variables  $\Phi$  and a diagonal matrix  $\tilde{S}$  commuting with  $N$  such that

$$\partial_\omega \Phi(\theta) = (S + N)\Phi(\theta) - \Phi(\theta)(\tilde{S} + N)$$

and every pair  $(\tilde{\lambda}_j, \tilde{\lambda}_k)$  of eigenvalues of  $\tilde{S}$  satisfies

$$i\langle v, \omega \rangle - \tilde{\lambda}_k + \tilde{\lambda}_l = 0 \Rightarrow i\langle v, \omega \rangle = 0 \quad (19)$$

*Proof.* Suppose there exist indices  $i_1, \dots, i_r$  and  $j_1, \dots, j_s$  such that  $\lambda_{i_1} = \dots = \lambda_{i_r}$ ,  $\lambda_{j_1} = \dots = \lambda_{j_s}$  and for some  $v$  such that  $\langle v, \omega \rangle \neq 0$ , one has  $\lambda_{i_1} - \lambda_{j_1} - i\langle v, \omega \rangle = 0$ . Let  $\Phi^1(\theta)$  be the diagonal matrix with diagonal coefficients  $(\Phi_1^1(\theta), \dots, \Phi_n^1(\theta))$  such that  $\Phi_{i_1}^1(\theta) = \dots = \Phi_{i_r}^1(\theta) = e^{i\langle v, \theta \rangle}$  and all other diagonal coefficients are equal to one. Then

$$\partial_\omega \Phi^1(\theta) = D \cdot \Phi^1(\theta)$$

where  $D$  is the diagonal matrix with coefficients  $D_{i_1, i_1} = \dots = D_{i_r, i_r} = i\langle v, \omega \rangle$  and all other coefficients equal to 0. By construction,  $\Phi^1$  commutes with  $S + N$  and so does  $D$ , therefore

$$\partial_\omega \Phi^1(\theta) = (S + N)\Phi^1(\theta) - \Phi^1(\theta)(S - D + N)$$

and the coefficients  $(\lambda_j^1)$  of  $S - D$  are the same as those of  $S$ , except that  $\lambda_{i_1}, \dots, \lambda_{i_r}$  are shifted to  $\lambda_{i_1}^1 = \lambda_{i_1} - i\langle v, \omega \rangle, \dots, \lambda_{i_r}^1 = \lambda_{i_r} - i\langle v, \omega \rangle$ , so that  $\lambda_{i_1}^1 = \dots = \lambda_{i_r}^1 = \lambda_{j_1}^1 = \dots = \lambda_{j_s}^1$ . Therefore, the change of variables has merged two groups of identical coefficients into one.

In the new matrix  $S - D$ , if another resonance appears, one can perform a similar change of variables in order to merge the two groups of coefficients which are in resonance with each other. Thus, in a finite number of steps (at most  $n - 1$ ), a new diagonal matrix  $\tilde{S}$  is obtained, which commutes with  $N$ , and in which every pair of coefficients are either identical or non-resonant with each other.  $\square$

From now on, we will assume that the eigenvalues of  $S$  satisfy (19). In the following statement, which gives the solution of the cohomological equation, recall that  $T^N F$  stands for a truncation that preserves the equivalence classes of Fourier modes.

**Proposition 4.6.** *Let  $S = \text{diag}(\lambda_j)$  be a constant Melnikov diagonal matrix. Let  $R(\theta) = S + M(\theta)$  where  $M(\theta)$  is resonant, upper triangular. Assume that for all  $1 \leq k, l \leq n$  and for all  $v \in \mathbb{Z}^d$ ,*

$$i\langle v, \omega \rangle - \lambda_k + \lambda_l = 0 \Rightarrow i\langle v, \omega \rangle = 0$$

and

$$|M|_r \leq 2\|A_0\| + 2\varepsilon_0$$

where  $\varepsilon_0 = \frac{\kappa'}{2n\|A_0\|^n}$ . Then for every  $N \geq 1$  and if  $F \in C_r^\omega$  is upper triangular, there is an upper triangular  $X \in C_r^\omega$  such that

$$\partial_\omega X = [R, X] + T^N F_{nr} \quad (20)$$

and

$$|X|_r \leq \frac{C''(n)(\|A_0\| + 1)}{\kappa'^{n+1}} N^d g'(N)^{n+1} |F|_r$$

where  $C''(n)$  only depends on  $n$ . Moreover, if  $T^N F_{nr}$  is upper triangular, then so is  $X$ .

*Proof.* Developing (20) into the subspaces  $\mathcal{H}_{\langle v \rangle}$  yields

$$(\mathcal{L}_{\langle v \rangle} + \mathcal{N}(\theta))X_{\langle v \rangle}(\theta) = T^N F_{\langle v \rangle}(\theta)$$

where  $\mathcal{L}_{\langle v \rangle} : gl(n, \mathbb{C}) \rightarrow gl(n, \mathbb{C})$ ,  $X \mapsto i\langle v, \omega \rangle X - [S, X]$  and  $\mathcal{N}(\theta) : X \mapsto -[M(\theta), X]$ . Since  $M$  is upper triangular, then  $M \in \mathcal{F}_1$ . According to the previous lemma,  $\mathcal{N}(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ . Hence, the operator  $\mathcal{N}$  is nilpotent.

If  $i\langle v, \omega \rangle - (\lambda_k - \lambda_l) \neq 0$  for all  $1 \leq k, l \leq n$ , then  $\mathcal{L}_{\langle v \rangle}$  is invertible and this equation amounts to

$$(Id + \mathcal{L}_{\langle v \rangle}^{-1} \mathcal{N}(\theta))X_{\langle v \rangle} = \mathcal{L}_{\langle v \rangle}^{-1} T^N F_{\langle v \rangle}(\theta) \quad (21)$$

One has the estimate

$$\|\mathcal{L}_{\langle v \rangle}^{-1}\| \leq \frac{C(n)}{\min_{j,k} |i\langle v, \omega \rangle + \lambda_j - \lambda_k|} \leq \frac{C(n)g'(|v|)}{\kappa'}$$

Each  $\mathcal{F}_k$  is left invariant by  $\mathcal{L}_{\langle v \rangle}$ . Therefore,  $\mathcal{L}_{\langle v \rangle}^{-1} \mathcal{N}(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ . Hence, the restriction to  $\mathcal{F}_k$  of  $\mathcal{L}_{\langle v \rangle}^{-1} \mathcal{N}$  is nilpotent. Hence, if  $T^N F_{\langle v \rangle}(\theta)$  belongs to  $\mathcal{F}_k$ , so does  $X_{\langle v \rangle}$  and there is a  $p \leq n$  such that  $(I + \mathcal{L}_{\langle v \rangle}^{-1} \mathcal{N}(\theta))^{-1} = \sum_{l=0}^p (\mathcal{L}_{\langle v \rangle}^{-1} \mathcal{N}(\theta))^l$ . Therefore,

$$|X_{\langle v \rangle}|_r \leq \sum_{l=0}^n \|\mathcal{L}_{\langle v \rangle}^{-1}\|^{l+1} |\mathcal{N}|_r^l |T^N F_{\langle v \rangle}|_r$$

where  $|\mathcal{N}|_r = \sup_{X \in C_r^\omega} \frac{|\mathcal{N}X|_r}{|X|_r}$  (so that  $|\mathcal{N}|_r \leq 2n|M|_r$ : recall that  $|M|_r = \sup_\theta |M(\theta)|$  since  $M$  is resonant). Let us set  $X_{res} := 0$  and  $X := X_{res} + \sum_{\langle v \rangle / \langle v, \omega \rangle \neq 0} X_{\langle v \rangle}$ ; then  $X$  is solution of (20) and

$$\begin{aligned}
|X|_r &\leq C(n) \sum_{\langle v \rangle} \sum_{l=0}^n \left( \frac{\kappa'}{g'(|v|)} \right)^{-(l+1)} |M|_r^l |T^N F_{\langle v \rangle}|_r \\
&\leq C'(n) |M|_r^n \frac{N^d g'(N)^{n+1}}{\kappa'^{n+1}} |F|_r \\
&\leq \frac{C''(n)(\|A_0\| + 1)}{\kappa'^{n+1}} N^d g'(N)^{n+1} |F|_r
\end{aligned} \tag{22}$$

□

**Proposition 4.7.** *Let  $S$  be a constant Melnikov diagonal matrix. Let  $R(\theta) = S + M(\theta)$  where  $M(\theta)$  is resonant, upper triangular. Let  $\varepsilon > 0$  and  $F \in C_r^\omega$  be upper triangular with  $|F|_r \leq \varepsilon$ . Let  $r' \in (0, r)$  be defined by*

$$\frac{|\ln \varepsilon|}{r - r'} = g''^{-1} \left( \frac{\kappa'^{n+1}}{C''(n)(\|A_0\| + 1)\sqrt{\varepsilon}} \right) \tag{23}$$

where  $g''(x) = x^d g'(x)^{n+1}$  and  $C''(n)$  is given by Proposition 4.6. Suppose also that

$$\varepsilon \leq \frac{1}{4C_N^2} (r - r')^{2d+2} \tag{24}$$

where  $C_N = N^d + dN^{d-1} + \dots + d!$ . Then there exists  $Z \in C_r^\omega$  such that

$$\partial_\omega Z = (R + F)Z - Z(R' + F')$$

where  $F'$  is upper triangular,  $R'(\theta) = S + M'(\theta)$  with  $M'$  resonant and upper triangular. We have  $|F'|_{r'} \leq \varepsilon^{\frac{3}{2}}$ , and  $|Z - I|_r \leq \sqrt{\varepsilon}$ .

*Proof.* Let  $N$  be such that  $\frac{1}{2}\varepsilon^{-\frac{1}{2}} = \frac{C''(n)(\|A_0\|+1)}{\kappa'^{n+1}} N^d g'(N)^{n+1}$ . By assumption (23) on  $r'$ ,  $e^{-N(r-r')} = \varepsilon$ . By Proposition 4.6, there exists  $X \in C_r^\omega$  such that  $\partial_\omega X = [R, X] + T^N F_{nr}$  and  $|X|_r < 1$ . Let  $Z = I + X$ . Then

$$\partial_\omega Z = [R, Z] + T^N F_{nr} = (R + F)Z - Z(R + Z^{-1}FZ - Z^{-1}T^N F_{nr})$$

(note that  $Z$  is invertible since  $|X|_r < 1$ .) Let  $R' = R + T^N F_{res}$ . Then

$$R + Z^{-1}FZ - Z^{-1}T^N F_{nr} = R' + Z^{-1}FZ - Z^{-1}T^N F_{nr} - T^N F_{res}$$

Let  $F' = Z^{-1}FZ - Z^{-1}T^N F_{nr} - T^N F_{res}$ ; thus

$$\begin{aligned}
F' &= \sum_{l \geq 0} (-X)^l F(I + X) - \sum_{l \geq 0} (-X)^l T^N F_{nr} - T^N F_{res} \\
&= F - T^N F + \sum_{l \geq 1} (-X)^l F(I + X) - \sum_{l \geq 1} (-X)^l T^N F_{nr} + FX
\end{aligned} \tag{25}$$

Now



$$|F - T^N F|_{r'} \leq \sum_{|k| > N} \|\hat{F}(k)\| e^{|k|r'} \leq \sum_{|k| > N} |F|_r e^{-|k|(r-r')} \leq \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')}$$

(where  $C_N = N^d + dN^{d-1} + \dots + d!$ ), thus

$$\begin{aligned} |F'|_{r'} &\leq \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')} + 6|F|_r |X|_r \\ &\leq \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')} + \frac{C''(n)(\|A_0\| + 1)}{\kappa'^{n+1}} N^d g'(N)^{n+1} |F|_r^2 \\ &\leq \frac{C_N |F|_r \varepsilon}{(r-r')^{d+1}} + \frac{1}{2} |F|_r^{\frac{3}{2}} \end{aligned} \quad (26)$$

(the last inequality holds because of the choice of the parameter  $N$ ). By the assumption (24), one has  $|F'|_{r'} \leq \varepsilon^{\frac{3}{2}}$ .

Moreover, if  $F$  is upper triangular, then so is  $X$ , thus  $F'$  as well since it is product of upper triangular matrices. As required,  $R'(\theta) = S + M'(\theta)$  where  $M'(\theta) = M(\theta) + F_{res}^N(\theta)$  is resonant and upper triangular.  $\square$

We need the following (trivial) lemma:

**Lemma 4.8.** *Let  $g''(x) = x^d g'(x)^{n+1}$ . If  $g''$  satisfies:*

$$\frac{|\ln \varepsilon|}{g''^{-1} \left( \frac{\kappa'^{n+1}}{C''(n)(\|A_0\| + 1)\sqrt{\varepsilon}} \right)} \geq (4C^2 \varepsilon)^{\frac{1}{2d+2}} \quad (27)$$

and if  $r' \in (0, r)$  is defined by

$$\frac{|\ln \varepsilon|}{r - r'} = g''^{-1} \left( \frac{\kappa'^{n+1}}{C''(n)(\|A_0\| + 1)\sqrt{\varepsilon}} \right) \quad (28)$$

then

$$\varepsilon \leq \frac{1}{4C^2} (r - r')^{2d+2} \quad (29)$$

The Brjuno-Rüssmann assumption on  $\omega$  gives a control on the loss of analyticity:

**Lemma 4.9.** *There exists  $\varepsilon_0 > 0$  which depends only on  $n, d, r, \kappa', g'$  such that if, for all  $k \in \mathbb{N}$ ,  $\varepsilon_k = \varepsilon_0^{(3/2)^k}$  and  $r_{k+1} = r_k - \frac{|\ln \varepsilon_k|}{g''^{-1} \left( \frac{\kappa'^{n+1}}{C''(n)(\|A_0\| + 1)\sqrt{\varepsilon_k}} \right)}$  with  $r_0 = r$ , then  $(r_k)$  has a positive limit  $r_\infty$ .*

*Proof.* One has

$$r - \sum_{k \geq 1} (r_k - r_{k+1}) = r - \sum_k \frac{|\ln \varepsilon_k|}{g''^{-1} \left( \frac{\kappa'^{n+1}}{C''(n)(\|A_0\| + 1)\sqrt{\varepsilon_k}} \right)}$$

and by the change of variables  $g''(Y) = \frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}$ ,

$$r - \sum_{k \geq 1} (r_k - r_{k+1}) \geq r - \int_{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_0}}\right)}^{\infty} \frac{C \ln g''(Y)}{Y} dY$$

where  $C = \frac{2\sqrt{\varepsilon_0}}{|\ln \varepsilon_0|}$ . Therefore  $r_\infty > 0$  if  $\varepsilon_0$  is small enough as a function of  $r, g'', \kappa', n, A_0$ . □

## 4.2 Proof of theorem 4.1

Let us define the following sequences: for all  $k \in \mathbb{N}$ ,

$$\varepsilon_k = \varepsilon_0^{\left(\frac{3}{2}\right)^k}$$

$$r_{k+1} = r_k - \frac{|\ln \varepsilon_k|}{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}\right)}$$

(by Lemma 4.9,  $r_k$  has a positive limit). Let  $N_k$  be defined by the relation

$$\frac{1}{2}\varepsilon_k^{-\frac{1}{2}} = \frac{C'''(n)(\|A_0\|+1)}{\kappa'^{n+1}} N_k^d g'(N_k)^{n+1}$$

The assumption (27) of Lemma 4.8 is satisfied, thus one can apply Proposition 4.7. Applying Proposition 4.7 with  $R$  is the constant map equal to  $S + N$ , one obtains  $Z \in C_{r_1}^\omega$  conjugating  $R + F$  to  $R_1 + F_1$  where

- $|F_1|_{r_1} \leq \varepsilon_1$ ,
- $R_1$  is resonant,  $R_1(\theta) = S + M_1(\theta)$  where  $M_1(\theta)$  is upper triangular,
- $F_1$  is upper triangular,
- $|Z - I|_{r_1} \leq \sqrt{\varepsilon_1}$ .

Now suppose, by induction, that  $A_0 + F$  is conjugated by  $Z_k \in C_{r_k}^\omega$  to  $R_k + F_k$  with the properties

- $|F_k|_{r_k} \leq \varepsilon_k$ ,
- $R_k(\theta) = S + M_k(\theta)$ , where  $M_k$  is resonant with upper triangular values,
- $F_k$  has upper triangular values,
- $|Z_k - I|_{r_k} \leq 2 \sum_{l=1}^k \sqrt{\varepsilon_l}$ .

Then applying again Proposition 4.7 (by means of Lemma 4.8), one has  $Z_{k+1} \in C_{r_{k+1}}^\omega$  conjugating  $A_0 + F$  to  $R_{k+1} + F_{k+1}$  with

- $|F_{k+1}|_{r_{k+1}} \leq \varepsilon_{k+1}$ ,
- $R_{k+1}(\theta) = S + M_{k+1}(\theta)$ , where  $M_{k+1}$  is resonant with upper triangular values,

- $F_{k+1}$  has upper triangular values,
- $|Z_{k+1} - I|_{r_{k+1}} \leq 2 \sum_{l=1}^{k+1} \sqrt{\varepsilon_l}$ .

Thus, for all  $k \in \mathbb{N}$ , the system  $A_0 + F$  is conjugated, in  $C_{\frac{r}{2}}^\omega$ , to  $R_k + F_k$ , where  $R_k$  is resonant,  $|F_k|_{\frac{r}{2}} \leq \varepsilon_k$ , and  $|Z_k - I|_{r_k} \leq 2$ .

By Lemma 4.9,  $r_k$  has a strictly positive limit  $r_\infty$ . Let  $R_\infty$  be a limit point, in  $C_{r_\infty}^\omega$ , of  $(R_k)_{k \in \mathbb{N}}$  (thus  $R_\infty$  is a resonant map); let  $(k_l)$  be a sequence such that  $R_{k_l}$  tends to  $R_\infty$ , and let  $Z_\infty$  be a limit point, in  $C_{r_\infty}^\omega$ , of the subsequence  $Z_{k_l}$ . Then  $\partial_\omega Z_\infty = (A_0 + F)Z_\infty - Z_\infty R_\infty$ . □

## 5 Analytic reduction of a strongly commuting perturbation

**Definition 5.1.** Let  $F \in C_r^\omega$ ;  $F$  is strongly commuting if for all equivalence classes  $v, v' \in \mathbb{Z}^d / \sim$ , one has  $[F_{\langle v \rangle}, F_{\langle v' \rangle}] = 0$ .

The aim of this section is to prove the following:

**Theorem 5.2.** Let  $F \in C_r^\omega$  be strongly commuting. Assume that there exists  $C > 0, 0 < R < r$  such that for all  $k \in \mathbb{Z}^d$  with  $\langle k, \omega \rangle \neq 0$ ,

$$|\langle k, \omega \rangle| \geq C e^{-|k|R} \quad (30)$$

Then the system with coefficient matrix  $F$  is analytically reducible to a resonant system.

*Proof.* The solution can be written as

$$X(t, \theta) = \exp \left( \int_0^t F(\theta + s\omega) ds \right) X(0, \theta)$$

The strong commutation assumption implies that the solution of the initial system can be directly computed and written in a reduced form:

$$\begin{aligned} X(t, \theta) &= e^{tF_{res}(\theta)} \prod_{\langle v \rangle \in \mathbb{Z}^d / \sim, \langle v, \omega \rangle \neq 0} \exp \left( \frac{e^{it\langle v, \omega \rangle} - 1}{i\langle v, \omega \rangle} \sum_{k \in \mathbb{Z}^d, k \in \langle v \rangle} \hat{F}_k e^{i\langle k, \theta \rangle} \right) \\ &= e^{tF_{res}(\theta)} \prod_{\langle v \rangle \in \mathbb{Z}^d / \sim, \langle v, \omega \rangle \neq 0} \exp \left( \frac{e^{it\langle v, \omega \rangle} - 1}{i\langle v, \omega \rangle} F_{\langle v \rangle}(\theta) \right) \\ &= Z(\theta + t\omega) e^{tF_{res}(\theta)} Z(\theta)^{-1} \end{aligned} \quad (31)$$

where  $Z(\theta) = \prod_{\langle v, \omega \rangle \neq 0} \exp \left( \frac{F_{\langle v \rangle}(\theta)}{i\langle v, \omega \rangle} \right)$ . Thus we seek an arithmetical condition on  $\omega$  under which  $Z$  is analytic. This will hold if the function

$$Y(\theta) := \sum_{\langle v, \omega \rangle \neq 0} \frac{F_{\langle v \rangle}(\theta)}{i\langle v, \omega \rangle}$$

is itself analytic. Now

$$Y(\theta) = \sum_{k \in \mathbb{Z}^d, \langle k, \omega \rangle \neq 0} \frac{\hat{F}(k)e^{i\langle k, \theta \rangle}}{i\langle k, \omega \rangle}$$

thus  $\hat{Y}(k) = \frac{\hat{F}(k)}{i\langle k, \omega \rangle}$  if  $\langle k, \omega \rangle \neq 0$ . Therefore  $Y$  is  $C_r^\omega$ -analytic if there is  $C > 0$  such that for all  $k$  with  $\langle k, \omega \rangle \neq 0$ ,

$$\|\hat{Y}(k)\| \leq Ce^{-|k|r'}$$

which holds if

$$|F|_r e^{-|k|(r-r')} \leq C|\langle k, \omega \rangle|.$$

Now this holds if  $\omega$  satisfies the condition (30) with  $R \leq r - r'$ .  $\square$

## 6 Preservation of Lie structures

Assuming that the initial system takes its values in a Lie algebra  $\mathfrak{g}$  among  $gl(n, \mathbb{R})$ ,  $sl(n, \mathbb{C})$  or  $sp(n, \mathbb{R})$  (for  $n$  even), a slight modification of the proofs will make the reduced system have its values in the same Lie algebra and the reducing transformation have its values in the corresponding Lie group  $G$ .

Firstly, notice that the homological equations (8) and (20) have a solution in the Lie algebra where the coefficients  $A, R$  and  $F$  have their values. This was used in [Cha13] (Proposition 2.8) in the non resonant case and works identically even if the frequency vector  $\omega$  is resonant, since by construction the solution is unique (since resonances are removed from the right-hand side of the homological equation). This comes from the fact that the Lie algebras considered here are defined by an equation of the form  $L(F) = 0$ , where  $L$  is a linear operator on matrices, and such that for all  $X$ , whenever  $A$  is in the Lie algebra, either  $L([A, X]) = 0$  (for instance if  $L$  is the trace operator) or  $L([A, X]) = [A, L(X)]$  (for instance if  $L(X) = J(X^*J + JX)$  where  $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ ).

Then one has to define a change of variables which takes its values in the Lie group. The change of variables defined above was  $I + X$ , one would have to take  $\exp(X)$  instead. Since, at the first order,  $I + X$  and  $\exp(X)$  coincide, the difference in the estimates will be quadratic, thus the new estimates will change only by a universal constant and will not prevent the convergence of the KAM scheme.

Also, the structure will be preserved by integration in the proofs of Proposition 3.2 and Theorem 5.2: integration preserves the zero trace property, commutes with the operator  $X \mapsto X^*J + JX$  and preserves the space of integrable real functions.

Therefore, Theorem 3.2 can be restated as follows:

**Theorem 6.1.** *Assume that  $\omega$  is very weak with exponent  $R > 0$  and that the eigenvalues of  $A_0$  have distinct real parts. Let  $r > R$ . Let  $A = A_0 + a$  be a  $C_r^\omega$ -analytic cocycle (1) with values in  $\mathfrak{g}$  where  $a$  is*

a zero-mean valued function satisfying  $|a|_r \leq \varepsilon_0(n, A_0)$ . Then there exists a convergent transformation with values in  $G$  conjugating (1) to a normal form in a neighborhood of the torus:

$$NF = \omega \frac{\partial}{\partial \theta} + D_{res}(\theta)x \frac{\partial}{\partial x}.$$

where  $D_{res}$  is a resonant diagonal matrix valued function with values in  $\mathfrak{g}$ .

and Theorem 5.2 becomes the following:

**Theorem 6.2.** *Assume that  $\omega$  is very weak with exponent  $R$ . Let  $r > R$ . Let  $A_0 \in \mathfrak{g}$ . Let  $F \in C_r^\omega$  be  $\mathfrak{g}$ -valued, strongly commuting and such that for all  $v, \theta$ ,  $[A_0, F_{\langle v \rangle}(\theta)] = 0$ . There exists  $C(A_0, n)$  such that if  $|F|_r \leq C$ , then the system  $A_0 + F$  is reducible, by a  $G$ -valued transformation, to a resonant system  $A_\infty(\theta)$  which commutes with  $A_0$  and has its values in  $\mathfrak{g}$ .*

In Section 4, it was assumed that the constant part  $A_0$  was in a Jordan normal form and it was said that this assumption is not restrictive. In order to remain in  $gl(n, \mathbb{R})$ , however, two more arguments are needed.

First of all, the Lie structure can slightly affect the way Lemma 4.5 is applied. Indeed, if  $S + N$  is the Jordan normal form of a real matrix, one will have to preserve the pairs of eigenvalues which are complex conjugate, and therefore it will be necessary to double the period. While doubling the period, new resonances might appear, namely  $\langle k, \omega \rangle$  with  $k \in \frac{1}{2}\mathbb{Z}^d$ . However, after doubling the period a finite number of times (at most  $n - 1$  times), all resonances will be deleted. So we will find the transformation not defined on the original torus but rather on a  $2^{n-1}$  covering.

While preserving the real structure, this way of eliminating the resonances also produces a transformation with determinant 1, thus the structure of  $SL(n, \mathbb{R})$  is also preserved.

On the other hand, let  $P$  be such that  $A_0 = P(S + N)P^{-1}$  where  $S$  is diagonal and  $N$  is nilpotent. Assume that  $PRP^{-1}$  and  $PF P^{-1}$  are real valued. If  $X$  is solution of (20), then  $PXP^{-1}$  is solution of  $\partial_\omega PXP^{-1} = [PRP^{-1}, PXP^{-1}] + PT^N F_{nr} P^{-1}$ , thus it is real. The change of variables  $Pe^X P^{-1}$  is thus also real, and its iteration is real (recall that taking  $e^X$  instead of  $I + X$  does not essentially change the estimates). The estimates are not changed except by the constant  $\|P^{-1}\| \cdot \|P\|$  since the change of basis is unchanged through the iteration. Finally, the system with constant part  $A_0$  can be reduced by a real-valued change of variables, and the reduced system is then automatically real.

Moreover, the trace is invariant by matrix conjugation. Thus, Theorem 4.1 can be restated as follows:

**Theorem 6.3.** *Let  $\mathfrak{g} = gl(n, \mathbb{R})$  or  $\mathfrak{g} = sl(n, \mathbb{R})$ . Let  $A_0 = P(S + M)P^{-1} \in \mathfrak{g}$  where  $S = \text{diag}(\lambda_j)$  is diagonal,  $M$  is nilpotent and upper triangular, and  $[M, S] = 0$ . Let  $F \in C_r^\omega(\mathbb{T}^d, \mathfrak{g})$  be such that  $P^{-1}FP$  is upper triangular valued. Assume that  $S$  is Melnikov and that  $|F|_r \leq \varepsilon_0(n, d, \kappa', g', A_0, r)$ . Then there exists an  $r_\infty > 0$  and a  $Z_\infty \in C_{r_\infty}^\omega(2^{n-1}\mathbb{T}^d, G)$  and a resonant  $R_\infty \in C_{r_\infty}^\omega(2^{n-1}\mathbb{T}^d, G)$ , such that  $P^{-1}RP$  has upper triangular values and such that  $\partial_\omega Z_\infty = (A_0 + F)Z_\infty - Z_\infty R_\infty$ .*

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