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# Inverse binomial series and values of Arakawa-Kaneko zeta functions

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## Abstract

In this article, we consider a variety of evaluations of series of polyarithmic nature. In particular, we express the special values at positive integers of certain zeta functions of Arakawa-Kaneko type in terms of series involving inverse binomial sums studied by Kalmykov and Davydychev in relation with the Feynman diagrams.

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## 1 Introduction

The function  $\beta$  defined for  $\Re(s) > 0$  by the Dirichlet series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

has the integral representation

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_0\left(\frac{1-e^{-2t}}{2}\right) t^{s-1} dt$$

where  $\operatorname{Li}_k$  denotes the classical polylogarithm  $\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ . One may also observe that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_1(1-e^{-2t}) t^{s-1} dt = (2-2^{-s})s \zeta(s+1).$$

We define two families of functions  $\alpha_k$  and  $\beta_k$  by the Mellin transforms

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k(1-e^{-2t}) t^{s-1} dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 1,$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1} dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 0,$$

so that

$$\alpha_1(s) = (2-2^{-s})s\zeta(s+1), \quad \text{and} \quad \beta_0(s) = \beta(s).$$

We notice a complete analogy between the couple of functions  $(\alpha_k, \beta_k)$  and the couple  $(\xi_k, \eta_k)$  defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(1-e^{-t}) t^{s-1} dt \quad \text{for } \Re(s) > 0, \quad k \geq 1,$$

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k\left(\frac{1-e^{-t}}{2}\right) t^{s-1} dt \quad \text{for } \Re(s) > 0, \quad k \geq 0,$$

where the function  $\xi_k$  was introduced by Arakawa and Kaneko (cf. [1], [4]). One can easily verify that

$$\xi_1(s) = s \zeta(s+1) = (2-2^{-s})^{-1}\alpha_1(s), \quad \text{and} \quad \eta_0(s) = (1-2^{1-s})\zeta(s).$$

The values  $\alpha_k(n)$  et  $\beta_k(n)$  where  $n$  is a positive integer can be expressed by means of the inverse binomial series studied by Kalmykov and Davydychev (cf. [5]) in relation with the Feynman diagrams. More precisely, we obtain the following identities:

$$\begin{aligned}
\alpha_1(1) &= \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{4}, \\
\beta_1(1) &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16}, \\
\alpha_2(1) &= \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{2} \ln 2 - \frac{7}{4} \zeta(3), \\
\beta_2(1) &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{16} \ln 2 + \frac{\pi}{2} G - \frac{35}{32} \zeta(3), \\
\alpha_3(1) &= \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^4} = \frac{\pi^2}{2} (\ln 2)^2 - \frac{7}{2} \zeta(3) \ln 2 + \frac{\pi^4}{96} + 4G(1), \\
\alpha_1(2) &= \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2} \zeta(3), \\
\beta_1(2) &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4} \zeta(3) - \frac{\pi}{2} G, \\
\alpha_2(2) &= \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^3} = 7\zeta(3) \ln 2 - \frac{\pi^4}{32} - 8G(1),
\end{aligned}$$

where we use the following notations:

$$O_n := \sum_{k=1}^n \frac{1}{2k-1} \text{ is the "odd" harmonic sum,}$$

$$G := \beta(2) \text{ is the Catalan constant,}$$

$$G(1) := \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \text{ is the Ramanujan constant (cf. [2] p. 257, [9]).}$$

The first two identities were known of Euler (cf. [8] p. 526), both following ones were known of Ramanujan (cf. [2] p. 269), and the four last ones seem to be new. The computation of  $\beta_1(n)$  is of particular interest, for it leads to another kind of new relations deduced by means of a series transformation (cf. section 4.1).

## 2 Bell polynomials and "odd" harmonic numbers

**Definition 1.** The *modified Bell polynomials* are the polynomials

$$P_m \in \mathbb{Q}[x_1, x_2, \dots, x_m]$$

defined for all natural numbers  $m$  by  $P_0 = 1$  and the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \dots, x_m) z^m,$$

The general explicit expression for  $P_m$  is

$$P_m(x_1, \dots, x_m) = \sum_{k_1+2k_2+\dots+mk_m=m} \frac{1}{k_1!k_2!\dots k_m!} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \dots \left(\frac{x_m}{m}\right)^{k_m}.$$

**Example 1.** For the first values of  $m$ , one has

$$\begin{aligned} P_0 &= 1, \\ P_1 &= x_1, \\ P_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2, \\ P_3 &= \frac{1}{6}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{3}x_3, \\ P_4 &= \frac{1}{24}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{1}{8}x_2^2 + \frac{1}{3}x_1x_3 + \frac{1}{4}x_4. \end{aligned}$$

**Notation.** For  $s \in \mathbb{C}$  with  $\Re(s) \geq 1$  and an integer  $n \geq 1$ , let  $O_n^{(s)}$  be the "odd" harmonic sum:

$$O_n^{(s)} = \sum_{k=1}^n \frac{1}{(2k-1)^s}, \quad \text{and} \quad O_n := O_n^{(1)}.$$

**Proposition 1.** For all integers  $m \geq 0$  and  $n \geq 1$ ,

$$P_m(O_n, \dots, O_n^{(m)}) = \frac{n \binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{-t} (1 - e^{-2t})^{n-1} \frac{t^m}{m!} dt. \quad (1)$$

*Proof.* We are going to prove that

$$\sum_{m=0}^{\infty} P_m(O_n, \dots, O_n^{(m)}) z^m = \prod_{j=1}^n \frac{2j-1}{2j-1-z} = \frac{n \binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz} (1 - e^{-2t})^{n-1} e^{-t} dt,$$

and then we shall obtain formula (1) by identification of the coefficients of  $z^m$ . On

one side, one has

$$\begin{aligned}
\prod_{j=1}^n \frac{2j-1}{2j-1-z} &= \prod_{j=1}^n \left(1 - \frac{z}{2j-1}\right)^{-1} \\
&= \exp\left(-\sum_{j=1}^n \log\left(1 - \frac{z}{2j-1}\right)\right) \\
&= \exp\left(\sum_{j=1}^n \sum_{k=1}^{+\infty} \frac{z^k}{k(2j-1)^k}\right) \\
&= \exp\left(\sum_{k=1}^{+\infty} \frac{z^k}{k} \sum_{j=1}^n \frac{1}{(2j-1)^k}\right),
\end{aligned}$$

thus

$$\prod_{j=1}^n \frac{2j-1}{2j-1-z} = \exp\left(\sum_{k=1}^{\infty} O_n^{(k)} \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(O_n, \dots, O_n^{(m)}) z^m.$$

On the other side, one has

$$\begin{aligned}
\prod_{j=1}^n \frac{2j-1}{2j-1-z} &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)} \\
&= \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)\Gamma(n)\Gamma(-z/2+1/2)}{\Gamma(n)\Gamma(n-z/2+1/2)} \\
&= \frac{n}{2^{2n}} \binom{2n}{n} B(n, -z/2+1/2),
\end{aligned}$$

where  $B$  is the Euler Beta function. Thus, for  $0 < |z| < 1$ , one has

$$\prod_{j=1}^n \frac{2j-1}{2j-1-z} = \frac{n \binom{2n}{n}}{2^{2n}} \int_0^1 u^{n-1} (1-u)^{-z/2-1/2} du,$$

and making the change of variable  $u = 1 - e^{-2t}$ , one then obtains:

$$\prod_{j=1}^n \frac{2j-1}{2j-1-z} = \frac{n \binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz} (1 - e^{-2t})^{n-1} e^{-t} dt,$$

and finally

$$\sum_{m=0}^{\infty} P_m(O_n, \dots, O_n^{(m)}) z^m = \frac{n \binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz} (1 - e^{-2t})^{n-1} e^{-t} dt.$$

□

### 3 The operators $D$ and $S$ , and the Euler series transformation

**Definition 2.** Let  $a$  be an analytic function in  $P = \{x \mid \Re(x) \geq 1\}$  defined by

$$a(x) = \int_0^{+\infty} e^{-xt} \widehat{a}(t) dt \quad \text{for all } x \in P,$$

where  $\widehat{a} \in \mathcal{C}^1([0, +\infty[)$  is such that there exists  $\alpha < 1$ , and  $C > 0$  with

$$|\widehat{a}(t)| \leq C e^{\alpha t} \quad \text{for all } t \in ]0, +\infty[.$$

For  $x \in P$ , we define the functions  $x \mapsto D(a)(x)$  and  $x \mapsto S(a)(x)$  by

$$D(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1 - e^{-t}} (1 - e^{-t})^x \widehat{a}(t) dt,$$

$$S(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1 - e^{-t}} (1 - e^{-xt}) \widehat{a}(t) dt.$$

**Proposition 2.** For all integers  $n \geq 1$ , one has

$$S(a)(n) = \sum_{k=1}^n a(k),$$

and for all integer  $n \geq 0$ ,

$$D(a)(n+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} a(k+1).$$

*Proof.* The first relation follows from

$$S(a)(n) = \int_0^{+\infty} \frac{e^{-t} - e^{-(n+1)t}}{1 - e^{-t}} \widehat{a}(t) dt = \int_0^{+\infty} \left( \sum_{k=1}^n e^{-kt} \right) \widehat{a}(t) dt = \sum_{k=1}^n \int_0^{+\infty} e^{-kt} \widehat{a}(t) dt.$$

The second relation results from the binomial expansion of  $(1 - e^{-t})^n$  since

$$\begin{aligned} D(a)(n+1) &= \int_0^{+\infty} e^{-t} (1 - e^{-t})^n \widehat{a}(t) dt \\ &= \int_0^{+\infty} e^{-t} \sum_{k=0}^n (-1)^k \binom{n}{k} e^{-kt} \widehat{a}(t) dt \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^{+\infty} e^{-t} e^{-kt} \widehat{a}(t) dt \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} a(k+1). \end{aligned}$$

□

**Example 2.** For  $s$  with  $\Re(s) \geq 1$  and  $x \in P$ , let

$$a(x) = \frac{1}{(2x-1)^s}.$$

One has

$$a(x) = \int_0^{+\infty} e^{-(2x-1)t} \frac{t^{s-1}}{\Gamma(s)} dt = \int_0^{+\infty} e^{-xt} \frac{e^{\frac{t}{2}} \left(\frac{t}{2}\right)^{s-1}}{2\Gamma(s)} dt.$$

Thus for all integer  $n \geq 1$ ,

$$D(a)(n) = \int_0^{+\infty} e^{-\frac{t}{2}} (1 - e^{-t})^{n-1} \frac{\left(\frac{t}{2}\right)^{s-1}}{2\Gamma(s)} dt = \int_0^{+\infty} e^{-t} (1 - e^{-2t})^{n-1} \frac{t^{s-1}}{\Gamma(s)} dt. \quad (2)$$

By Proposition 1 above, one also has

$$\int_0^{+\infty} e^{-t} (1 - e^{-2t})^{n-1} \frac{t^m}{m!} dt = \frac{2^{2n-1}}{n \binom{2n}{n}} P_m(O_n, \dots, O_n^{(m)}).$$

Thus, if  $s$  is an integer,  $s = m + 1$  with  $m \geq 0$ , then we get for all integers  $n \geq 1$  the following formula

$$D\left(\frac{1}{(2x-1)^{m+1}}\right)(n) = \frac{2^{2n-1}}{n \binom{2n}{n}} P_m(O_n, \dots, O_n^{(m)}). \quad (3)$$

**Lemma 1.** The operators  $D$  and  $S$  are linked by the following relation:

$$D\left(\frac{1}{x}S(a)\right) = \frac{1}{x}D(a) \text{ for all } x \in P.$$

*Proof of the lemma.* By definition of  $S(a)$ ,

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} \frac{1 - e^{-xt}}{x} \left[ \frac{e^{-t}}{1 - e^{-t}} \hat{a}(t) \right] dt,$$

integrating by parts, we get

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} e^{-xt} \left( \int_t^{\infty} \frac{e^{-u}}{1 - e^{-u}} \hat{a}(u) du \right) dt,$$

this gives

$$\widehat{\frac{1}{x}S(a)}(t) = \int_t^{\infty} \frac{e^{-u}}{1 - e^{-u}} \hat{a}(u) du.$$



Thus

$$D\left(\frac{1}{x}S(a)\right)(x) = \int_0^{+\infty} e^{-t}(1 - e^{-t})^{x-1} \left( \int_t^{\infty} \frac{e^{-u}}{1 - e^{-u}} \widehat{a}(u) du \right) dt,$$

and integrating again by parts, we get

$$D\left(\frac{1}{x}S(a)\right)(x) = \int_0^{+\infty} \frac{1}{x}(1 - e^{-t})^x \frac{e^{-t}}{1 - e^{-t}} \widehat{a}(t) dt = \frac{1}{x}D(a)(x).$$

□

**Proposition 3.** For all complex numbers  $z$  such that  $|z| < \frac{1}{2}$ , one has

$$\sum_{n=1}^{+\infty} D(a)(n)z^n = - \sum_{n=1}^{+\infty} a(n) \left(\frac{z}{z-1}\right)^n, \quad (4)$$

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = - \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) \left(\frac{z}{z-1}\right)^n. \quad (5)$$

*Proof.* For the first relation we write

$$\begin{aligned} \sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= \int_0^{+\infty} e^{-t} \frac{z}{1 - (1 - e^{-t})z} \widehat{a}(t) dt \\ &= - \int_0^{+\infty} \left(\frac{z}{z-1}\right) \frac{e^{-t}}{1 - e^{-t} \frac{z}{z-1}} \widehat{a}(t) dt. \end{aligned}$$

The expansion

$$\left(\frac{z}{z-1}\right) \frac{e^{-t}}{1 - e^{-t} \frac{z}{z-1}} \widehat{a}(t) = \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{z}{z-1}\right)^n \widehat{a}(t)$$

gives

$$\begin{aligned} \sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= - \int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{z}{z-1}\right)^n \widehat{a}(t) dt \\ &= - \sum_{n=1}^{+\infty} \left(\frac{z}{z-1}\right)^n \int_0^{+\infty} e^{-nt} \widehat{a}(t) dt, \end{aligned}$$

the order of  $\int_0^{+\infty}$  and  $\sum_{n=1}^{\infty}$  may be interchanged because

$$\int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{|z|}{1 - |z|}\right)^n |\widehat{a}(t)| dt = \left(\frac{|z|}{1 - |z|}\right) \int_0^{+\infty} \frac{e^{-t}}{1 - e^{-t} \frac{|z|}{1 - |z|}} |\widehat{a}(t)| dt < +\infty.$$

The second relation is an immediate consequence of the first one by Lemma 1 above. □

**Proposition 4.** For all integers  $p \geq 1$ , one has

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \text{Li}_k\left(\frac{1-e^{-t}}{p}\right) \widehat{a}(t) dt.$$

*Proof.* Let  $p$  be a positive integer, then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} &= \sum_{n=1}^{\infty} \frac{1}{p^n n^k} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} (1-e^{-t})^n \widehat{a}(t) dt \\ &= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k} \widehat{a}(t) dt \\ &= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \text{Li}_k\left(\frac{1-e^{-t}}{p}\right) \widehat{a}(t) dt, \end{aligned}$$

the order of  $\int_0^{+\infty}$  and  $\sum_{n=1}^{\infty}$  may be interchanged since, by the hypothesis on  $\widehat{a}$ ,

$$|\widehat{a}(t)| \leq C e^{\alpha t} \text{ for all } t \in ]0, +\infty[ ,$$

which gives

$$\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k} |\widehat{a}(t)| dt \leq C \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \text{Li}_k\left(\frac{1-e^{-t}}{p}\right) e^{\alpha t} dt < +\infty.$$

□

## 4 The functions $\alpha_k$ and $\beta_k$

**Definition 3.** Let  $k$  be a positive integer. The functions  $\alpha_k$  and  $\beta_k$  are respectively defined for all  $s \in \mathbb{C}$  with  $\Re(s) > 0$  by

$$\begin{aligned} \alpha_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \text{Li}_k(1-e^{-2t}) t^{s-1} dt \quad (\text{for } k \geq 1), \\ \beta_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \text{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1} dt \quad (\text{for } k \geq 0). \end{aligned}$$

**Example 3.**

$$\begin{aligned} \alpha_1(s) &= \frac{2}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} t^s dt = 2(1-2^{-s-1}) s \zeta(s+1), \\ \beta_1(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} [\ln 2 - \ln(1+e^{-2t})] t^{s-1} dt. \end{aligned}$$

**Proposition 5.** If  $s$  is such that  $\Re(s) \geq 1$ , then

$$\alpha_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{(2x-1)^s}\right)(n) \quad (\text{for } k \geq 1),$$

$$\beta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D\left(\frac{1}{(2x-1)^s}\right)(n) \quad (\text{for } k \geq 0).$$

*Proof.* This is an immediate consequence of Proposition 4 applied to the function  $a(x) = \frac{1}{(2x-1)^s}$  (for  $p = 1, 2$ ) since  $\hat{a}(t) = e^{\frac{t}{2}} \frac{\left(\frac{t}{2}\right)^{s-1}}{2\Gamma(s)}$  as already seen in Example 2.  $\square$

**Corollary 1.** For all integers  $m \geq 0$ , then

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \dots, O_n^{(m)}) \quad (\text{for } k \geq 1), \quad (6)$$

$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \dots, O_n^{(m)}) \quad (\text{for } k \geq 0). \quad (7)$$

*Proof.* This is an immediate consequence of Proposition 5 and formula (3).  $\square$

**Example 4.** Since  $\alpha_1(s) = (2 - 2^{-s})s\zeta(s+1)$ , then for all integers  $m \geq 1$ ,

$$(2 - 2^{-m})m\zeta(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \dots, O_n^{(m-1)}). \quad (8)$$

In particular, for  $m = 1$ , this gives

$$\zeta(3) = \frac{1}{7} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{n^2}. \quad (9)$$

**Example 5.** Since  $\beta_0(s) = \beta(s)$ , one has

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n}{n} = 2G, \quad (10)$$

and for all integers  $m \geq 1$ ,

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n} n} P_{2m}(O_n^{(1)}, O_n^{(2)}, \dots, O_n^{(2m)}) = \frac{E_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m+1}, \quad (11)$$

where  $E_{2m}$  ( $m = 1, 2, \dots$ ) are the Euler numbers (cf. [8] 11 (c) p. 544) defined by the generating function:

$$\frac{1}{\cos z} = 1 + \sum_{m=1}^{\infty} E_{2m} \frac{z^{2m}}{(2m)!} = 1 + \frac{z^2}{2!} + \frac{5z^4}{4!} + \frac{61z^6}{6!} + \dots .$$

In particular, for  $m = 1$ , this gives

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{n} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{n} = \frac{\pi^3}{8} = 4\beta(3) .$$

**Remark 1.** In a similar way (cf. [3] § 5.5), one can prove for the couple of functions  $(\xi_k, \eta_k)$  the following identities:

$$\begin{aligned} \xi_k(s) &= \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^s}\right)(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 1), \\ \eta_k(s) &= \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D\left(\frac{1}{x^s}\right)(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 0), \end{aligned}$$

and, furthermore, one also has (cf. [3] § 3)

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n^{(1)}, \dots, H_n^{(m)})}{n} \quad \text{with } H_n^{(j)} = \sum_{k=1}^n \frac{1}{k^j} \quad (j = 1, 2, \dots, m) .$$

This gives for example

$$\begin{aligned} \xi_1(2) &= \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \\ \eta_1(2) &= \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 \quad ([2] \text{ p. 258}) . \end{aligned}$$

## 4.1 The function $\beta_1$

**Proposition 6.** For all  $s \in \mathbb{C}$  with  $\Re(s) \geq 1$ , one has

$$\beta_1(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(s)}}{n},$$

hence, for each integer  $m \geq 1$ ,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \dots, O_n^{(m-1)}) . \quad (12)$$

*Proof.* By Proposition 3 above, we have for all  $|z| < \frac{1}{2}$ ,

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = - \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) \left(\frac{z}{z-1}\right)^n.$$

If the series  $\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n}$  is convergent, then, by the classical Abel lemma, we get

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n} = \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (-1)^{n-1}.$$

By Proposition 4 above, the series  $\sum_{n=1}^{\infty} \frac{1}{2^n n} D\left(\frac{1}{(2x-1)^s}\right)(n)$  is convergent and

$$\beta_1(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n} D\left(\frac{1}{(2x-1)^s}\right)(n) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} S\left(\frac{1}{(2x-1)^s}\right)(n).$$

Then, using formula (3) for  $D\left(\frac{1}{(2x-1)^m}\right)(n)$ , one obtains (12).  $\square$

**Example 6.**

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n} &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16}, \\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(2)}}{n} &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4} \zeta(3) - \frac{\pi}{2} G \end{aligned} \quad (13)$$

([5] (2.36) and (2.37) with  $u = 2$  and  $\theta = \frac{\pi}{2}$ ),

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(3)}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{(2n)^2} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{(2n)^2} = \frac{\pi^4}{64} - G^2 \quad (14)$$

([5] (2.38), (2.39), (2.40) and (C.4) with  $u = 2$  and  $\theta = \frac{\pi}{2}$ ).

**Remark 2.** By Remark 1 above, we get also the relation

$$\eta_1(m) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} P_{m-1}(H_n^{(1)}, H_n^{(2)}, \dots, H_n^{(m-1)}) \quad (15)$$

which is similar to (12). In particular, one has the following relation:

$$\eta_1(2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2.$$

**Proposition 7.** Let

$$\widetilde{H}_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{k}.$$

If  $s$  is such that  $\Re(s) > 1$ , then

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^s} = (1-2^{-s})\zeta(s) \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)^s} - \beta_1(s). \quad (16)$$

Thus, for each integer  $m > 1$ ,

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^m} = (1-2^{-m})\zeta(m) \ln 2 + (-1)^m \ln 2 + 2 \sum_{k=1}^m (-1)^{m-k} \beta(k) - \beta_1(m). \quad (17)$$

*Proof.* The first relation is a direct consequence of the following elementary result:

If the series  $\sum_{n=1}^{\infty} a_n$ ,  $\sum_{n=1}^{\infty} b_n$ ,  $\sum_{n=1}^{\infty} a_n b_n$  and  $\sum_{n=1}^{\infty} b_n \sum_{k=1}^n a_k$  are convergent, then the series  $\sum_{n \geq 1} a_n \sum_{k=1}^n b_k$  is convergent and we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^n b_k = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n b_n - \sum_{n=1}^{\infty} b_n \sum_{k=1}^n a_k.$$

Applied to  $a_n = \frac{1}{(2n-1)^s}$  and  $b_n = \frac{(-1)^{n-1}}{n}$ , this relation gives (16). The second relation is a consequence of the first one by the following observation:

If  $s$  is an integer,  $s = m$ , then we have

$$\sum_{k=1}^m \frac{(-1)^k}{(2n-1)^k} = \frac{(-1)^m (2n-1)^{-m} - 1}{2n},$$

hence

$$\frac{(-1)^{n-1}}{n(2n-1)^m} = (-1)^m \frac{(-1)^{n-1}}{n} + 2 \sum_{k=1}^m (-1)^{m-k} \frac{(-1)^{n-1}}{(2n-1)^k}$$

which gives formula (17). □

**Example 7.** Formula (17) gives respectively for  $m = 2$  and  $m = 3$  the following identities:

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^2} = \frac{\pi^2}{8} \ln 2 + \ln 2 - \frac{\pi}{2} + 2G - \frac{7}{4}\zeta(3) + \frac{\pi G}{2}, \quad (18)$$

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^3} = \frac{7}{8}\zeta(3) \ln 2 - \ln 2 + \frac{\pi}{2} - 2G + \frac{\pi^3}{16} + G^2 - \frac{\pi^4}{64}. \quad (19)$$

## 5 The values $\alpha_k(1)$ and $\beta_k(1)$

**Proposition 8.** For each integer  $k \geq 1$  and  $\alpha \in \mathbb{R}$  such that  $0 \leq \alpha \leq 1$ , let  $L_k(\alpha)$  be the (generalized) log-sine integral

$$L_k(\alpha) = \int_0^{\alpha\pi} u \ln^{k-1} \left( 2 \sin \frac{u}{2} \right) du = \pi^2 \int_0^\alpha x \ln^{k-1} \left( 2 \sin \frac{\pi x}{2} \right) dx.$$

Then, one has

$$2\alpha_k(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = 2^{k-1} \sum_{i=1}^k (-1)^{i-1} \frac{(\ln 2)^{k-i}}{(i-1)!(k-i)!} L_i(1), \quad (20)$$

$$2\beta_k(1) = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{1}{n^{k+1}} = \sum_{i=1}^k (-1)^{i-1} \frac{2^{i-1} (\ln 2)^{k-i}}{(i-1)!(k-i)!} L_i\left(\frac{1}{2}\right). \quad (21)$$

*Proof.* The proof is similar to that given in [10] § 4. Let

$$J_k(x) = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n} n^{k+1}}.$$

Then, one has for  $k \geq 1$ ,

$$J_k(x) = \int_0^x \frac{J_{k-1}(u)}{u} du.$$

By a classical identity due to Euler (cf. [8] p. 526, [10]), one also has

$$J_1(x) = (\arcsin x)^2,$$

hence

$$J_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}}.$$

It is easily verified that

$$2\alpha_k(1) = 2^k J_k(1) = 2^k \int_0^1 \frac{J_{k-1}(x)}{x} dx,$$

and

$$2\beta_k(1) = 2^k J_k\left(\frac{\sqrt{2}}{2}\right) = 2^k \int_0^{\frac{\sqrt{2}}{2}} \frac{J_{k-1}(x)}{x} dx.$$

By  $(k-1)$  integrations by parts and the change of variable  $x = \sin \frac{u}{2}$ , we get

$$J_k(1) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \ln^{k-1}(x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^\pi u \ln^{k-1} \left( \sin \frac{u}{2} \right) du,$$

and

$$J_k\left(\frac{\sqrt{2}}{2}\right) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^{\frac{\sqrt{2}}{2}} \ln^{k-1}(\sqrt{2}x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\frac{1}{2}\pi} u \ln^{k-1}(\sqrt{2} \sin \frac{u}{2}) du.$$

It remains to use the binomial expansions of

$$\ln^{k-1}\left(\sin \frac{u}{2}\right) = \left[ \ln\left(\frac{1}{2}\right) + \ln\left(2 \sin \frac{u}{2}\right) \right]^{k-1},$$

and

$$\ln^{k-1}(\sqrt{2} \sin \frac{u}{2}) = \left[ \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(2 \sin \frac{u}{2}\right) \right]^{k-1}$$

to obtain formulas (20) and (21). □

**Proposition 9.** For all  $\alpha$  such that  $0 \leq \alpha \leq 1$ , we have

$$\begin{aligned} \text{a) } L_1(\alpha) &= \frac{\pi^2}{2} \alpha^2, \\ \text{b) } L_2(\alpha) &= \zeta(3) - \sum_{n=1}^{\infty} \frac{\cos(\pi n \alpha)}{n^3} - \alpha \pi \sum_{n=1}^{\infty} \frac{\sin(\pi n \alpha)}{n^2}, \\ \text{c) } L_3(\alpha) &= \frac{\pi^4}{16} \left( \alpha^4 - \frac{8}{3} \alpha^3 + 2 \alpha^2 \right) + 2 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} \cos(\pi(n+1)\alpha) \\ &\quad + 2\pi\alpha \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \sin(\pi(n+1)\alpha) - \frac{1}{2} \zeta(4). \end{aligned} \tag{22}$$

*Proof.* The assertion a) is trivially verified and b) is a classical identity (cf. [6], formula (7.53)). It remains to prove c). We use the following expansion :

$$\text{Log}^2(1-z) = 2 \sum_{n=1}^{\infty} H_n \frac{z^{n+1}}{n+1}$$

to get

$$\text{Log}^2\left(1 - e^{-i\pi x}\right) = 2 \sum_{n=1}^{\infty} H_n \frac{e^{-i\pi x(n+1)}}{n+1}.$$

Since

$$\begin{aligned} \text{Log}^2\left(1 - e^{-i\pi x}\right) &= \text{Log}^2\left(e^{-i\pi x/2} (e^{i\pi x/2} - e^{-i\pi x/2})\right) \\ &= \left(-i\pi x/2 + i\pi/2 + \ln\left(2 \sin \frac{1}{2} x \pi\right)\right)^2 \\ &= -\frac{\pi^2}{4} (x-1)^2 + \ln^2\left(2 \sin \frac{\pi x}{2}\right) + i\Im\left(\text{Log}^2\left(1 - e^{-i\pi x}\right)\right), \end{aligned}$$



one has

$$\ln^2\left(2 \sin \frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + \Re\left(\text{Log}^2\left(1 - e^{-i\pi x}\right)\right),$$

hence

$$\ln^2\left(2 \sin \frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + 2 \sum_{n=1}^{\infty} H_n \frac{\cos(\pi(n+1)x)}{n+1}.$$

Integrating, this gives

$$\begin{aligned} \int_0^\alpha x \ln^2\left(2 \sin \frac{\pi x}{2}\right) dx &= \frac{\pi^2}{4} \int_0^\alpha x(x-1)^2 dx \\ &\quad + 2 \sum_{n=1}^{\infty} \frac{H_n}{n+1} \int_0^\alpha x \cos(\pi(n+1)x) dx. \end{aligned} \quad (23)$$

The permutation of  $\sum$  and  $\int$  in (23) is justified by the following Lemma 2 and the dominated convergence theorem. The integrals in the right-hand side of (23) are easily computed by

$$\int_0^\alpha x(x-1)^2 dx = \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2},$$

and

$$\int_0^\alpha x \cos(\pi(n+1)x) dx = \frac{\cos(\pi(n+1)\alpha)}{\pi^2(n+1)^2} + \frac{\alpha}{\pi(n+1)} \sin(\pi(n+1)\alpha) - \frac{1}{\pi^2(n+1)^2}.$$

Thus, we deduce from (23) the following expression for  $L_3(\alpha)$ :

$$\begin{aligned} L_3(\alpha) &= \frac{\pi^4}{4} \left( \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2} \right) + 2 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} \cos(\pi(n+1)\alpha) \\ &\quad + 2\pi\alpha \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \sin(\pi(n+1)\alpha) - 2 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3}. \end{aligned}$$

Moreover, one has

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} = \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \zeta(4) = \frac{5}{4}\zeta(4) - \zeta(4) = \frac{1}{4}\zeta(4),$$

and this gives (22). □

**Lemma 2.** The partial sums

$$\sum_{n=1}^k H_n \frac{x \cos(\pi(n+1)x)}{n+1}$$

are uniformly bounded for  $x \in ]0, 1[$ .

*Proof of the lemma.* Let  $S_n(x) = x \sum_{j=1}^n \cos(\pi(j+1)x)$ . A sommation by parts gives

$$\sum_{n=1}^k \frac{H_n}{n+1} x \cos(\pi(n+1)x) = \sum_{n=1}^k S_n(x) \left( \frac{H_n}{n+1} - \frac{H_{n+1}}{n+2} \right) + \frac{H_k}{k+1} S_k(x),$$

and one has

$$|S_n(x)| = \left| x \sum_{j=1}^n \cos(\pi(j+1)x) \right| \leq \frac{2x}{\sin(\pi x/2)}.$$

It follows that, for all  $x \in ]0, 1[$ ,

$$\left| \sum_{n=1}^k x H_n \frac{\cos(\pi(n+1)x)}{n+1} \right| \leq \frac{2x}{\sin(\pi x/2)} \left( \frac{H_1}{2} - \frac{H_{k+1}}{k+2} + \frac{H_k}{k+1} \right) \leq \frac{Cx}{\sin(\pi x/2)} \leq C'.$$

□

**Example 8.** Formulae (20) and (21) give for  $k = 2$  the following relations:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^3} &= \pi^2 \ln 2 - \frac{7}{2} \zeta(3), \\ \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{1}{n^3} &= \frac{\pi^2}{8} \ln 2 + \pi G - \frac{35}{16} \zeta(3). \end{aligned}$$

These two identities were known of Ramanujan (cf. [2], p. 269).

## 6 New formulae for Ramanujan's constant $G(1)$

In Chapter 9 of his notebooks (cf. [2] p. 255, Entry 11), Ramanujan introduced the two generating functions<sup>1</sup>

$$F(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^2} \quad \text{and} \quad G(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^3},$$

then, he writes the following functional relation :

$$\begin{aligned} G(x) + G\left(\frac{1-x}{1+x}\right) &= F(x) \log(x) + F\left(\frac{1-x}{1+x}\right) \log\left(\frac{1-x}{1+x}\right) \\ &\quad - \frac{1}{16} \log^2(x) \log^2\left(\frac{1-x}{1+x}\right) + C, \end{aligned} \tag{ii}$$

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<sup>1</sup>These functions are respectively quoted  $\phi$  and  $\psi$  in the original manuscript : cf. [7] p. 108.

with

$$C = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)^3} - \frac{\pi}{3\sqrt{3}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}.$$

Unfortunately, this beautiful formula found by Ramanujan for  $C$  turns out to be erroneous since the constant  $C$  in (ii) must be equal to

$$G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \quad (\text{cf. [2] p. 257, or [9] for more details}).$$

However, we now show how the calculation of  $\alpha_3(1)$  and  $\alpha_2(2)$  provides two interesting formulae for the constant  $G(1)$ .

**Proposition 10.** Let  $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$  be the Ramanujan constant. One has

$$G(1) = \frac{7}{8} \zeta(3) \ln 2 - \frac{\pi^4}{384} - \frac{1}{8} \pi^2 (\ln 2)^2 + 2 \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{(2n)^4}. \quad (24)$$

*Proof.* First, we prove the two following identities:

$$G(1) = \frac{35}{64} \zeta(4) - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3}, \quad (25)$$

and

$$L_3(1) = \frac{\pi^4}{96} + 4G(1). \quad (26)$$

One has

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} &= \sum_{n=1}^{\infty} \frac{H_{2n} - \frac{1}{2}H_n}{(2n)^3} \\ &= \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} - \sum_{n=1}^{\infty} \frac{1}{2} \frac{H_n}{(2n)^3} \\ &= \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{2} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{1}{2^4} \frac{H_n}{n^3} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^3} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} - \frac{1}{2^4} \sum_{n=1}^{\infty} \frac{H_n}{n^3} \\ &= \frac{35}{64} \zeta(4) - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3}. \end{aligned}$$

Moreover, by Proposition 9 c) with  $\alpha = 1$ , one also has

$$L_3(1) = \frac{25}{8} \zeta(4) - 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} = \frac{\pi^4}{96} + 4G(1).$$

Then, applying Proposition 8, formula (20), it results from (26) that

$$2\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^4} = \pi^2(\ln 2)^2 - 7\zeta(3) \ln 2 + \frac{\pi^4}{48} + 8G(1). \quad (27)$$

□

**Remark 3.** We have seen before (cf. Example 4, formula (9)) that

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1} O_n}{\binom{2n}{n} n^2} = \frac{7}{2}\zeta(3),$$

and one also knows (cf. [2], p. 259) that

$$4F(1) = \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3).$$

Thus

$$F(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^2}. \quad (28)$$

The calculation of  $\alpha_2(2)$  provides a nice expression of the Ramanujan constant  $G(1)$  similar to (28).

**Proposition 11.** Let  $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$  be the Ramanujan constant. One has the following formula:

$$G(1) = \frac{7}{8}\zeta(3) \ln 2 - \frac{\pi^4}{256} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}. \quad (29)$$

*Proof.* Applying (2.44) and (2.45) of [5] with  $u = 4$  and  $\theta = \pi$ , one obtains an expression of  $\alpha_2(2)$  involving  $L_3(1)$ . By means of (26) previously established, one can simplify this expression to obtain

$$\alpha_2(2) = 7\zeta(3) \ln 2 - \frac{\pi^4}{32} - 8G(1) \quad (30)$$

which is equivalent to (29). □

**Remark 4.** Since  $\frac{7}{8}\zeta(3) = \sum_{n \geq 0} \frac{1}{(2n+1)^3}$  and  $\frac{\pi^3}{32} = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^3}$ , formula (29) may be rewritten

$$G(1) = 2 \ln(2) \sum_{n \geq 0} \frac{1}{(4n+1)^3} - (\pi/8 + \ln(2)) \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}$$

which gets closer to the mysterious erroneous formula given by Ramanujan for  $C$ .

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