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Thin r -neighborhoods of embedded geodesics with finite length and negative Jacobi operator are strongly convex*

Ph. Delanoë†

Abstract

In a complete Riemannian manifold, an embedded geodesic γ with finite length and negative Jacobi operator admits an r -neighborhood $N_r(\gamma)$ with radius $r > 0$ small enough such that each couple of points of $N_r(\gamma)$ can be joined by a unique geodesic contained in $N_r(\gamma)$ where it minimizes length among the piecewise C^1 paths joining its end points.

Introduction

Let M be a connected complete Riemannian manifold; let d denote its Riemannian distance function [dC92]. A connected subset $S \subset M$ with non empty interior $\overset{\circ}{S}$ is called strongly convex for a couple of points $(p, q) \in S \times S$ if there exists a unique geodesic path $t \in [0, 1] \rightarrow \gamma(t) \in M$ such that: $\gamma(0) = p, \gamma(1) = q$ and $\gamma(t) \in \overset{\circ}{S}$ for $t \in (0, 1)$, with γ length minimizing among piecewise C^1 paths from p to q in $\overset{\circ}{S}$. The subset S is just called *strongly convex* if it is so for each couple $(p, q) \in S \times S$.

Definition 0.1 *Let $S \subset M$ be a strongly convex subset. For each couple $(p, q) \in S \times S$, the length of the geodesic path joining p to q with interior in $\overset{\circ}{S}$ is called the inner distance from p to q in S , denoted by $d_S(p, q)$.*

It is quite natural to endow a strongly convex subset $S \subset M$ with its inner distance function d_S . The latter is nothing but the length metric associated with the metric space $(S, d|_S)$ [Gro99].

Since Whitehead's landmark paper [Whi32], it has been known that small enough balls in M are strongly convex. Moreover, if B is such a ball, its inner distance function d_B coincides with the restriction of d to $B \times B$ [KN96, CE08, Aub98, dC92, Kli95]. In the flat torus $\mathbb{R}^n/\mathbb{Z}^n$, if the radius of

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a ball B belongs to the interval $(\frac{1}{4}, \frac{1}{2})$, the reader can check that B remains strongly convex but d_B no longer coincides with $d|_{B \times B}$. Here, we would like to construct a general family of examples of strongly convex subsets $S \subset M$ such that $d_S \neq d|_{S \times S}$.

The notion of *extended* distance function used in [FRV12] is similar in spirit to that of inner metric; could it guide us toward an example? Let us recall its definition. If $t \in [0, 1] \rightarrow \gamma(t) \in M$ is an embedded geodesic without conjugate points, the map $\text{Id} \times \exp : TM \rightarrow M \times M$ induces a diffeomorphism Ψ_γ from a neighborhood \mathcal{U} of $(\gamma(0), \frac{d\gamma}{dt}(0))$ in TM to a neighborhood \mathcal{W} of $(\gamma(0), \gamma(1))$ in $M \times M$. The extended distance function d_γ of [FRV12] is then defined in \mathcal{W} by $d_\gamma(p, q) = |V|_p$ where $\Psi_\gamma(p, V) = (p, q)$. It is called so because, if γ contains no cut point, shrinking \mathcal{W} if necessary, it satisfies $d_\gamma(p, q) \equiv d(p, q)$. In this setting, we would like to know whether a thin enough tube about the geodesic γ must be strongly convex. Anytime it is, one may identify d_γ with the restriction to \mathcal{W} of the inner distance function of the tube; in particular, the function d_γ satisfies in effect the distance axioms.

By a *tube* about γ is meant a closed subset of M containing $\gamma([0, 1])$, with non empty interior and each point of which admitting a unique *nearest point* in $\gamma([0, 1])$; moreover, if $p \mapsto p_\gamma^\perp$ denotes the nearest point map, the geodesic from p to p_γ^\perp should meet $\gamma([0, 1])$ *orthogonally*. Finally, the lateral boundary of the tube is given by the equation $d(p, p_\gamma^\perp) = r$, where $r > 0$ is a small real number called the *radius* of the tube.

We are thus willing to study the question: *under which conditions must a tube about an embedded geodesic be strongly convex?*

First of all, indeed, we should restrict to geodesics without conjugate points (at least in their interior) since, by the Morse Index Theorem, they would not be minimizing otherwise [Mil63]. To proceed further, let us take examples. In the domain of the unit sphere of \mathbb{R}^3 given by: $0 \leq \text{longitude} < \pi$ and $-r \leq \text{latitude} \leq r$ with r small, we see that the geodesic joining two points with equal latitude close enough to r does not stay in that domain. But if we look at a similar domain about the *interior* equator of a torus of revolution in \mathbb{R}^3 and pick two points as above, the geodesic joining them does stay in the domain. So, a curvature assumption should be made along a geodesic before we can expect the strong convexity of a tube about it, and positive curvature rules out strong convexity.

Eventually, we will show that a tube $T_r(\gamma_0)$ with small enough radius r about a geodesic γ_0 with negative Jacobi operator is essentially strongly convex. Specifically, we will prove the following result:

Theorem 0.1 *Let $\gamma_0 : s \in [0, \ell_0] \rightarrow \gamma_0(s) \in M$ be an embedded unit speed geodesic with negative Jacobi operator. Given $\varsigma > 0$, there exists $\varrho > 0$ such that, if $r \in (0, \varrho)$, the tube $T_r(\gamma_0)$ is strongly convex for each couple $(p, q) \in T_r(\gamma_0) \times T_r(\gamma_0)$ of points satisfying, either $|s(p_\gamma^\perp) - s(q_\gamma^\perp)| \geq \varsigma$ or,*

$s(p_{\gamma_0}^\perp)$ and $s(q_{\gamma_0}^\perp)$ belong to the subinterval $[\varsigma, \ell_0 - \varsigma]$. Furthermore, if M has dimension 2, the result holds with $\varsigma = 0$ provided we except the boundary couples (p, q) lying in the same end ($s = 0$ or $s = \ell_0$) of the tube.

In this statement, we allow the geodesic γ_0 to contain cut points. For instance, if the image of γ_0 is contained in the curve $\{x^2 + y^2 = 1, z = 0\}$ viewed as the interior equator of a torus of revolution in \mathbb{R}^3 , we allow its length ℓ_0 to belong to the interval $[0, 2\pi)$. In this context, the inner distance function which we are looking for appears well approximated by the pseudo-metric defined in the tube by: $\widehat{d}(p, q) = |s(p_{\gamma_0}^\perp) - s(q_{\gamma_0}^\perp)|$, at least for the couples $(p, q) \in T_r(\gamma_0) \times T_r(\gamma_0)$ such that $\widehat{d}(p, q) \gg r$. Accordingly, our proof will split in two parts; let us provide a rough outline of it.

Case 1: for $\widehat{d}(p, q)$ less than a suitable positive constant c independent of r as $r \downarrow 0$, there exists a unique minimizing geodesic $t \in [0, 1] \rightarrow \gamma(t) \in M$ from p to q , so we only have to prove the inclusion $\gamma((0, 1)) \subset \widehat{T_r(\gamma_0)}$. We do it using a one parameter family of geodesics $\lambda \in [0, 1] \rightarrow c_\lambda$ interpolating between c_0 given by $t \in [0, 1] \rightarrow \gamma_0(ts(q_{\gamma_0}^\perp) + (1-t)s(p_{\gamma_0}^\perp))$ and $c_1 = \gamma$. For λ small, we certainly have $c_\lambda((0, 1)) \subset \widehat{T_r(\gamma_0)}$. We must rule out the possibility that $c_\lambda(t)$ first touches the boundary of $T_r(\gamma_0)$ for some $t \in (0, 1)$. If $n = 2$, it could happen but on the *lateral* part of $\partial T_r(\gamma_0)$ because the ends of $T_r(\gamma_0)$ are *totally geodesic*. If $n > 2$, the pinching $s[(c_\lambda(t))_{\gamma_0}^\perp] \in (0, \ell_0)$ is obtained relying on the assumption¹ $\widehat{d}(p, q) \geq \varsigma$ or $s(p_{\gamma_0}^\perp)$ and $s(q_{\gamma_0}^\perp)$ lie in $[\varsigma, \ell_0 - \varsigma]$. As for the lateral part of $T_r(\gamma_0)$, the estimate $d(c_\lambda(t), (c_\lambda(t))_{\gamma_0}^\perp) < r$ (unless $p = q$) follows from a *Maximum Principle* for geodesics shown to hold in $T_r(\gamma_0)$ due to our curvature assumption.

Case 2: $\widehat{d}(p, q) \geq c$. Here, we must work harder, shrink $r > 0$ and show that, if $t \in [0, 1] \rightarrow \gamma(t) \in M$ is a geodesic from p to q ranging in $T_r(\gamma_0)$, its Jacobi operator should stay, like the one of γ_0 , *negative*. Moreover, we infer from the latter property that γ must be minimizing and unique. We are thus left with proving the very *existence* of γ . It will be done by a tricky connectedness argument, fixing p , letting q vary in the tube and using the parameter $z = \widehat{d}(p, q) \in [c, \ell_0]$ itself. The openness part of that argument is based on the invertibility of $d(\exp_p)(\dot{\gamma}(0))$, which holds due to the curvature property of γ ; the closedness part relies on the aforementioned Maximum Principle.

Can one find a quicker proof? We did not. With Theorem 0.1 and its proof at hand, it becomes easy to obtain a full strong convexity result if, instead of the tube $T_r(\gamma_0)$, we consider the closure of the r -neighborhood of γ_0 , that is the subset $N_r(\gamma_0) = \{m \in M, d(\gamma_0([0, \ell_0]), m) \leq r\}$. In this way, we get the main result of the paper, namely:

¹ignored elsewhere in the proof

Corollary 0.1 (main result) *Let $\gamma_0 : s \in [0, \ell_0] \rightarrow \gamma_0(s) \in M$ be an embedded unit speed geodesic with negative Jacobi operator. There exists $\varrho > 0$ such that the subset $N_r(\gamma_0) \subset M$ is strongly convex for $r \in (0, \varrho)$.*

The paper is organized as follows: the next two sections are devoted to preliminary tools for the proof, general properties of thin tubes are recorded in Section 1 and further ones under our curvature assumption in Section 2; the proof of Theorem 0.1 itself is given in Section 3, that of Corollary 0.1, in Section 4.

1 Properties of a thin tube about an embedded geodesic

Throughout this section, we use the setting of Theorem 0.1 but drop the assumption made on the Jacobi operator of the geodesic γ_0 .

1.1 Fermi map, cylinders and Gauss Lemma

Let us recall how the tube $T_r(\gamma_0)$ can be precisely defined [Aub98, Gra04]. The geodesic γ_0 extends uniquely as a geodesic embedding of an interval $I = (-\epsilon, \ell_0 + \epsilon)$ with ϵ small. We consider the map:

$$(V, s) \in V_0^\perp \times I \longrightarrow E_0(V, s) = \exp_{\gamma_0(s)}^\perp(\|^\gamma(V)) \in M,$$

where we have denoted by V_0^\perp the subspace of $T_{\gamma_0(0)}M$ orthogonal to the velocity vector $V_0 = \frac{d\gamma_0}{ds}(0)$, by $\|^\gamma(V)$ the vector field along γ_0 obtained by parallel transport of the vector V and by $\exp_{\gamma_0(s)}^\perp$ the restriction of the exponential map to $\|^\gamma(V_0)(s)^\perp$. The differential of E_0 at $(0, s)$ is given by:

$$(\delta s, \delta V) \in V_0^\perp \times \mathbb{R} \rightarrow dE_0(0, s)(\delta V, \delta s) = \frac{d\gamma_0}{ds}(s)\delta s + \|^\gamma(\delta V)(s) \in T_{\gamma_0(s)}M;$$

it is an isomorphism since orthogonality is preserved by parallel transport. From the inverse function theorem [Lan02] and the compactness of $[0, \ell_0]$ (or bounded length of γ_0), we infer² the existence of a real $R > 0$ such that, setting $|V|$ for the norm of a vector V and $\overline{B}^\perp(0, R) = \{V \in V_0^\perp, |V| \leq R\}$, the map E_0 induces a *diffeomorphism* from a neighborhood of $\overline{B}^\perp(0, R) \times [0, \ell_0]$ onto a neighborhood of its image. Let us fix such a radius R once for all. For $r \leq R$, we denote by $T_r(\gamma_0)$ the image by E_0 of $\overline{B}^\perp(0, r) \times [0, \ell_0]$ and call it the *tube* about γ_0 with radius r [Gra04]. We set $p \mapsto F_0(p) = (v_0^\perp(p), z(p))$ for the *inverse* of the mapping E_0 and refer to it as the *Fermi map along γ_0* . We call $z(p)$ the *height* of the point p relative to

²full details are given in Section 1.4 for a construction encompassing the present one

γ_0 and the subsets $E_R^{\text{top}}(\gamma_0) = \{p \in T_R(\gamma_0), z(p) = \ell_0\}$ and $E_R^{\text{bot}}(\gamma_0) = \{p \in T_R(\gamma_0), z(p) = 0\}$ respectively for the *top and bottom ends* of the tube. If $p \in T_R(\gamma_0)$, the unit speed geodesic $s \in [0, |v_0^\perp(p)|] \rightarrow E_0\left(s \frac{v_0^\perp(p)}{|v_0^\perp(p)|}, z(p)\right)$ is the unique minimizing geodesic from γ_0 to p ; its length $\mathfrak{r}_{\gamma_0}(p) = |v_0^\perp(p)|$ is thus equal to $d(\gamma_0, p)$. For short, that geodesic will be denoted by $s \mapsto [\gamma_0, p](s) \in T_R(\gamma_0)$, and the function \mathfrak{r}_{γ_0} itself, simply by \mathfrak{r} unless a confusion may occur. We set $N_{\gamma_0}(p)$, or just $N(p)$ if no confusion, for the velocity vector $\frac{d[\gamma_0, p]}{ds}$ evaluated at $s = d(\gamma_0, p)$. The unit vector field $p \mapsto N(p)$ is defined in the open subset of the tube $T_R(\gamma_0)$ where $\mathfrak{r}(p) > 0$, that is, outside the geodesic γ_0 ; moreover, it is readily seen to satisfy $dz(N) = 0$, $d\mathfrak{r}(N) = 1$ and $\nabla_N N = 0$, with ∇ the Levi-Civita connection. If $r \in (0, R]$, we set $C_r(\gamma_0) = \{p \in T_R(\gamma_0), \mathfrak{r}(p) = r\}$ for the *cylinder* of radius r about γ_0 , sometimes called the *lateral part* of the boundary of the tube $T_r(\gamma_0)$. The outward unit *normal* to that cylinder at $p \in C_r(\gamma_0)$ is nothing but $N(p)$ due to the *generalized Gauss lemma* according to which the gradient of the function \mathfrak{r} and the vector field N coincide [Gra04, pp.26–28]. The identity $N = \text{grad } \mathfrak{r}$ will be central for us. It yields the following identity, recorded here for later use, valid at each $p \in T_R(\gamma_0)$ such that $\mathfrak{r}(p) > 0$:

$$(1) \quad \forall (V, W) \in T_p M \times T_p M, (g - d\mathfrak{r}^2)(V, W) = (g - d\mathfrak{r}^2)(\Pi_N^\perp(V), \Pi_N^\perp(W)),$$

where we have set $\Pi_N^\perp(V) = V - g(V, N)N$ for the orthogonal projection of $T_p M$ onto $N(p)^\perp$; in other words, if we write $TM = \mathbb{R}N \oplus N^\perp$ on $\{\mathfrak{r} > 0\}$, the generalized Gauss lemma implies that the metric g splits into the sum of $d\mathfrak{r}^2$ along $\mathbb{R}N$ and $(g - d\mathfrak{r}^2)$ along N^\perp .

Finally, $\mathfrak{i} \in (0, \infty]$ will stand for the injectivity radius of $T_R(\gamma_0)$, that is for the minimum of the distance from a point p to its cut locus as p varies in $T_R(\gamma_0)$ [dC92, pp.267–273]. For each $r \in (0, R]$, the injectivity radius of $T_r(\gamma_0)$ will thus be *at least equal* to \mathfrak{i} . If M is compact, \mathfrak{i} is finite; but $\mathfrak{i} = \infty$ if M is the hyperbolic space, for instance.

1.2 Fermi charts and related notions

Given an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_{\gamma_0(0)}M$ with $e_n = \frac{d\gamma_0}{ds}(0)$ (henceforth, n stands for the dimension of M), let us assign to each $p \in T_R(\gamma_0)$ the n -tuple $x = (\tilde{x}, x^n) \in \overline{B}^{n-1}(0, R) \times [0, \ell_0]$, where $\overline{B}^{n-1}(0, R)$ denotes the closure of the ball of radius R in the Euclidean space \mathbb{R}^{n-1} , given by:

$$x(p) = (x^1, \dots, x^{n-1}, x^n) \text{ if and only if } v_0^\perp(p) = \sum_{\alpha=1}^{n-1} x^\alpha e_\alpha \text{ and } z(p) = x^n.$$

The map $x : T_R(\gamma_0) \rightarrow \overline{B}^{n-1}(0, R) \times [0, \ell_0]$ so defined is called a *Fermi chart* along the embedded geodesic γ_0 (in 1922, while a PhD student at the Scuola Normale Superiore in Pisa, motivated by the study of the equivalence principle in

general relativity, Enrico Fermi was the first to consider such local coordinates, which he used along timelike paths, see [GV82, p.217] and references therein).

We see from this construction that $y = (\tilde{y}, y^n)$ is another such chart if and only if $y^n = x^n$ and there exists an orthogonal transformation $\mathfrak{R} \in O(n-1)$ such that $\tilde{y} = \mathfrak{R}\tilde{x}$. The calculations which we will perform in the tube $T_R(\gamma_0)$ will be *invariant* (or tensorial) with respect to change of Fermi charts. We will freely use the local Euclidean metric $\epsilon_{\gamma_0} = \sum_{i=1}^n (dx^i)^2$ (just denoted by ϵ , unless confusing) and the *affine structure* inherited from its (flat) Levi-Civita connection $D_{\gamma_0} = D$. The latter will be convenient to identify distinct tangent spaces hence view vectors tangent to $T_R(\gamma_0)$ at distinct points as belonging to *the same* vector space. We will also view the Christoffel symbols $\Gamma_{ij}^k(x)$ of our original (global) connection ∇ as the components in the chart x of the local *tensorial* difference $(\nabla - D)$.

In the Fermi chart x , the components of the metric tensor g satisfy: $g_{ij}(0, x^n) = \delta_{ij}$, $dg_{ij}(0, x^n) = 0$, so the Christoffel symbols vanish at $(0, x^n)$, meaning that g is osculating to ϵ along γ_0 . We set $\|\cdot\|$ for the norm associated to the Euclidean metric ϵ and $\theta_0 = \min \|U\| \leq 1 \leq \Theta_0 = \max \|U\|$, where U runs over all unit³ tangent vectors at points of $T_R(\gamma_0)$. For each $p \in T_R(\gamma_0)$, setting $\rho(x) = \sqrt{\sum_{\alpha=1}^{n-1} (x^\alpha)^2}$, we have $\mathfrak{r}(p) = \rho(x(p))$. The geodesic ray $t \in [0, 1] \rightarrow E_0(tv_0^\perp(p), z(p)) \in M$ reads $t \mapsto \mathcal{R}(t) = (tx^1, \dots, tx^{n-1}, x^n)$ with $x = x(p)$; being constant, its speed is equal to $\rho(x)$, so the unit vector field N reads: $N(p) = \nu(x(p))$ with $\nu(x) = \frac{1}{\rho(x)} \sum_{\alpha=1}^{n-1} x^\alpha \frac{\partial}{\partial x^\alpha}$.

If $W = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i} \in T_p M$, we may view W as a *constant* vector field in $T_R(\gamma_0)$, in other words extend it to $T_R(\gamma_0)$ by D_{γ_0} parallelism, a notion well defined in any Fermi chart along γ_0 . Following [Gra04, p.21], let us call any such vector field a *Fermi field* (here, with respect to γ_0). Given a point $p \in T_R(\gamma_0)$ and vector field Z on $T_R(\gamma_0)$, we may similarly consider the Fermi field $Z(p)$, thinking of it as Z frozen at p . Among Fermi fields, one may distinguish those with $W^n = 0$ from those writing $Z = Z^n \frac{\partial}{\partial x^n}$ (sometimes called *axial*). For later use, we record the brackets identities:

$$(2) \quad \forall \alpha < n, \quad \left[\nu, \rho \frac{\partial}{\partial x^\alpha} \right] = \frac{\partial \rho}{\partial x^\alpha} \nu, \quad \text{and} \quad \left[\nu, \frac{\partial}{\partial x^n} \right] = 0.$$

Finally, it will be convenient to consider on $T_R(\gamma_0)$ the field of projections $\Pi_0 = \sum_{\alpha=1}^{n-1} dx^\alpha \otimes \frac{\partial}{\partial x^\alpha}$, which is the constant (or Fermi) extension of the orthogonal projection of $T_{\gamma_0(0)} M$ onto V_0^\perp .

³here and below, to be understood for the metric g , unless otherwise specified

1.3 Estimates for geodesics in a thin tube

Beforehand, let us recall a classical result, namely: there exists a continuous function $p \in M \rightarrow \chi(p) \in (0, \infty]$ called the *convexity radius*, which is smaller than the injectivity radius, such that, for each $\varrho \in (0, \chi(p))$, the Riemannian ball $B(p, \varrho)$ is strongly convex [CE08, pp.103–105] [Kli95, pp.84–85] [Whi32]. For $r > 0$ small, we may thus consider the function $r \mapsto \chi_{\gamma_0}(r) = \min\{\chi(p), p \in T_r(\gamma_0)\}$, which is non increasing. We set $\mathfrak{c} = \chi_{\gamma_0}(R)$ and stress that $\mathfrak{c} \leq \mathfrak{i}$. Our first estimate is an upper bound on the length of the geodesics contained in the tube $T_{R_0}(\gamma_0)$ with $R_0 = \min\left(R, \frac{\mathfrak{c}}{3}\right)$.

Proposition 1.1 *If $\gamma : [0, \ell] \rightarrow \gamma(s) \in T_{R_0}(\gamma_0)$ is a unit speed geodesic⁴, its length ℓ is bounded above by L_0 , with: $L_0 = \ell_0 + 2R$ if $\mathfrak{i} = \infty$, and $L_0 = 2(\ell_0 + \mathfrak{c})$ if $\mathfrak{c} < \infty$.*

Proof. If $\mathfrak{i} = \infty$, the geodesic γ is minimizing and unique in M . But we can join its endpoints $p = \gamma(0), q = \gamma(\ell)$ by a geodesic path broken twice, namely, first by going along the geodesic ray from p to $\gamma_0(z(p))$, next by going from $\gamma_0(z(p))$ to $\gamma_0(z(q))$ along γ_0 , then by going along the geodesic ray from $\gamma_0(z(q))$ to q . The total length of that broken path must be larger than ℓ and it is, indeed, at most equal to $L_0 = \ell_0 + 2R$.

If $\mathfrak{c} < \infty$, for each $\epsilon > 0$ small enough, the triangle inequality satisfied by the Riemannian distance on M shows that we can cover the tube $T_{R_0}(\gamma_0)$ by N open balls of radius $r = \mathfrak{c} - \epsilon$, successively centered at the points $\gamma_0(0), \gamma_0(r), \gamma_0(2r), \dots, \gamma_0((N-1)r), \gamma_0(\ell_0)$, with $N = \left\lceil \frac{\ell_0}{\mathfrak{c}} \right\rceil + 1$. Now, the length of the restriction of the geodesic γ to each ball is bounded above by $2r$ and, letting $\epsilon \downarrow 0$, we obtain $\ell \leq 2N\mathfrak{c}$ \square

Using a Fermi chart along γ_0 , setting $R_1 = \frac{9}{10}R_0$, we can readily find a positive constant c_1 such that, for each $p \in T_{R_1}(\gamma_0)$, the following estimates hold at $x = x(p)$:

$$(3) \quad \|g - \mathfrak{e}\| \leq c_1 \rho^2(x), \quad \|\nabla - D\| \leq c_1 \rho(x).$$

The purpose of our next proposition is twofold. On the one hand, it provides a radius under which the geodesics contained in a tube about γ_0 and longer than a given length $\delta > 0$ keep moving axially in a single direction; in particular, they must be embedded, like γ_0 . On the other hand, it provides an estimate describing how C^0 close to γ_0 a geodesic should be in order to get C^1 close to it.

Proposition 1.2 *Fixing $\delta \in (0, L_0)$, let $r_1 > 0$ be given by:*

$$r_1^2 \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\delta} + c_1 L_0 \Theta_0^2 \right)^2 \right) = 1.$$

⁴Throughout the paper, ℓ denotes the length of γ which may vary; it should be written $\ell(\gamma)$, of course, but we will stick to the short notation ℓ instead.

For each $r \in (0, \min(R_1, r_1))$ and each unit speed geodesic $s \in [0, \ell] \rightarrow \gamma(s) \in T_r(\gamma_0)$ with length $\ell \geq \delta$, the axial component $\frac{d\gamma^n}{ds}$ of the velocity cannot vanish. Moreover, the following estimate holds:

$$\left\| \varepsilon \frac{d\gamma}{ds} - \frac{\partial}{\partial x^n} \right\| \leq \left(\frac{4}{\ell} + c_1 \ell \Theta_0^2 \right) \rho_\gamma + \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\ell} + c_1 \ell \Theta_0^2 \right)^2 \right) \rho_\gamma^2,$$

where ρ_γ stands for $\max_{\sigma \in [0, \ell]} \rho(\gamma(\sigma))$ and $\varepsilon = \pm 1$, for the sign of $\frac{d\gamma^n}{ds}$.

Proof. Before proving the first assertion we require an estimate, namely, letting $s \in [0, \ell] \rightarrow \gamma(s) \in T_{R_1}(\gamma_0)$ be a unit speed geodesic, we have:

$$(4) \quad \forall s \in [0, \ell], \left\| \Pi_0 \frac{d\gamma}{ds}(s) \right\| \leq \left(\frac{4}{\ell} + c_1 \ell \Theta_0^2 \right) \rho_\gamma.$$

Indeed, if $s \in [0, \frac{\ell}{2}]$, we write: $\forall \alpha \in \{1, \dots, n-1\}$,

$$(\ell - s) \frac{d\gamma^\alpha}{ds}(s) = \gamma^\alpha(\ell) - \gamma^\alpha(s) - \int_s^\ell \int_s^S \frac{d^2 \gamma^\alpha}{d\sigma^2}(\sigma) d\sigma dS;$$

while if $s \in [\frac{\ell}{2}, \ell]$, we write instead:

$$s \frac{d\gamma^\alpha}{ds}(s) = \gamma^\alpha(s) - \gamma^\alpha(0) - \int_0^s \int_s^S \frac{d^2 \gamma^\alpha}{d\sigma^2}(\sigma) d\sigma dS.$$

In either case, transforming the last term of the right-hand side by means of the geodesic equation, recalling (3) and using the triangle and Schwarz inequalities, we readily infer (4). Writing:

$$\left| \frac{d\gamma^n}{ds} \right| = \left\| \frac{d\gamma}{ds} \right\| \sqrt{1 - \frac{\left\| \Pi_0 \frac{d\gamma}{ds} \right\|^2}{\left\| \frac{d\gamma}{ds} \right\|^2}} \quad \text{and} \quad \left\| \frac{d\gamma}{ds} \right\| = \sqrt{1 - (g - \mathbf{e}) \left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right)},$$

the latter to be combined with (3), we get $\left| \frac{d\gamma^n}{ds} \right| \geq 1 - c_1 \rho_\gamma^2 \Theta_0^2 - \frac{1}{\theta_0^2} \left\| \Pi_0 \frac{d\gamma}{ds} \right\|^2$ hence, using (4), we obtain the important lower bound:

$$(5) \quad \forall s \in [0, \ell], \left| \frac{d\gamma^n}{ds}(s) \right| \geq 1 - \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\ell} + c_1 \ell \Theta_0^2 \right)^2 \right) \rho_\gamma^2.$$

Recalling Proposition 1.1, and the assumption $\ell \geq \delta$, it shows that $\frac{d\gamma^n}{ds}$ cannot vanish provided the radius r of the tube in which the geodesic ranges satisfies: $r^2 \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\delta} + c_1 L_0 \Theta_0^2 \right)^2 \right) < 1$, or else $r \in (0, \min(R_1, r_1))$, as we assumed. The first part of Proposition 1.2 is thus proved.

Moreover, letting now ε stand for the *sign* of $\frac{d\gamma^n}{ds}$, we have $\left| \frac{d\gamma^n}{ds}(s) \right| \equiv \varepsilon \frac{d\gamma^n}{ds}$, so we readily get from (5) and the obvious inequality $\left| \frac{d\gamma^n}{ds} \right| \leq \left\| \frac{d\gamma}{ds} \right\|$, the pinching:

$$-\frac{1}{2}c_1\Theta_0^2\rho_\gamma^2 \leq 1 - \varepsilon \frac{d\gamma^n}{ds} \leq \left(c_1\Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\ell} + c_1\ell\Theta_0^2 \right)^2 \right) \rho_\gamma^2.$$

Combined with (4), it yields the estimate claimed in the second part of Proposition 1.2, since $\left\| \varepsilon \frac{d\gamma}{ds}(s) - \frac{\partial}{\partial x^n} \right\| \leq \left\| \Pi_0 \frac{d\gamma}{ds}(s) \right\| + \left| \varepsilon \frac{d\gamma^n}{ds}(s) - 1 \right|$ \square

Setting UM for the unit tangent bundle and $\text{End}_s(TM)$ for the bundle of symmetric⁵ endomorphisms of TM , let us consider the map:

$$(p, U) \in UM \rightarrow \mathbb{J}(p, U) = R_p(\cdot, U)U \in \text{End}_s(TM),$$

where R_p stands for the Riemann curvature tensor at the point $p \in M$. It satisfies $g(V, \mathbb{J}(p, U)W) \equiv S_p(V, U, W, U)$ where S_p stands for the sectional (or covariant Riemann) curvature tensor of the metric g at the point p ; it is thus, indeed, *symmetric*. We denote by $\kappa^1(p, U) \leq \dots \leq \kappa^{n-1}(p, U)$ the eigenvalues (each repeated with its multiplicity) of the nontrivial part of $\mathbb{J}(p, U)$, namely of its restriction to U^\perp . For each $\alpha \in \{1, \dots, n-1\}$, the map $(p, U) \in UM \rightarrow \kappa^\alpha(p, U) \in \mathbb{R}$ is C_{loc}^1 [Kat95, pp.122–123], hence uniformly Lipschitz for $p \in T_{R_0}(\gamma_0)$. So there exists a constant k_0 such that, for each couple $(p, U), (p', U') \in UM^2$ with $\max(\mathfrak{r}_{\gamma_0}(p), \mathfrak{r}_{\gamma_0}(p')) \leq R_0$ and each $\alpha \in \{1, \dots, n-1\}$, the following uniform estimate holds:

$$(6) \quad \left| \kappa^\alpha(p, U) - \kappa^\alpha(p', U') \right| \leq k_0(d(p, p') + \|U - U'\|).$$

For each unit speed geodesic $\sigma \in [0, \ell] \rightarrow \gamma(\sigma) \in M$, we set $s \mapsto \mathfrak{J}_\gamma(s)$ for the pull back to $[0, \ell]$ of the map \mathbb{J} by the section $t \mapsto \left(\gamma(s), \frac{d\gamma}{ds}(s) \right) \in UM$ and call $\mathfrak{J}_\gamma(s)$ the *Jacobi operator* along the geodesic γ at s . We further set $\kappa_\gamma^1(s) \leq \dots \leq \kappa_\gamma^{n-1}(s)$ for the eigenvalues of the restriction of $\mathfrak{J}_\gamma(s)$ to $\frac{d\gamma}{ds}(s)^\perp$ and call them the *Jacobi curvatures* along γ at s .

Corollary 1.1 *Given δ and r as in Proposition 1.2, set:*

$$k = k_0 \left(1 + \frac{4}{\delta} + c_1 L_0 \Theta_0^2 + \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\delta} + c_1 L_0 \Theta_0^2 \right)^2 \right) r \right).$$

For each unit speed geodesic $\sigma \in [0, \ell] \rightarrow \gamma(\sigma) \in T_r(\gamma_0)$ with length $\ell \geq \delta$ and each $s \in [0, \ell]$, the following estimate holds:

$$(7) \quad \forall \alpha \in \{1, \dots, n-1\}, \quad \left| \kappa_\gamma^\alpha(s) - \kappa_0^\alpha(\gamma^n(s)) \right| \leq k \rho_\gamma,$$

where $\kappa_0^1 \leq \dots \leq \kappa_0^{n-1}$ stands for the Jacobi curvatures along γ_0 .

⁵here, 'unit' and 'symmetric' refer to the Riemannian metric g , of course

Proof. Indeed, fixing γ as stated, we may apply Proposition 1.2 to it. It yields an estimate on $\left\| \frac{d\gamma}{d\sigma}(s) - \frac{\partial}{\partial x^n} \right\|$ which, combined with the estimate (6) read at: $(p, U) = (\gamma(s), \frac{d\gamma}{d\sigma}(s))$ and $(p', U') = (\gamma_0(\gamma^n(s)), \frac{\partial}{\partial x^n})$, yields the desired result \square

Corollary 1.1 shows in particular that, if the Jacobi operator along γ_0 stays *definite*, it must stay so (with the same signature) along geodesics longer than a given length and contained in a tube about γ_0 of small enough radius.

1.4 Family of Fermi maps near γ_0

For each unit speed geodesic $s \in [0, \ell] \rightarrow \gamma(s) \in T_{R_1}(\gamma_0)$, let $I_{\gamma_0}(\gamma) \subset [0, \ell_0]$ denote the axial image interval $\gamma^n([0, \ell])$ and $T(\gamma_0, \gamma)$, the shortest piece of tube about γ_0 containing γ , equal to $\{m \in T_{\rho_\gamma}(\gamma_0), x^n(m) \in I_{\gamma_0}(\gamma)\}$. If such a geodesic γ is an embedding, when is it possible to construct a Fermi map along it such that a point $m \in T(\gamma_0, \gamma)$ may stay *outside* the corresponding tube about γ if and only if its height $z_\gamma(m)$ relative to γ satisfies either $z_\gamma(m) < 0$ or $z_\gamma(m) > \ell$? When such a possibility occurs, we call (γ_0, γ) -*exceptional* the latter points and (γ_0, γ) -*accessible* all other points of $T(\gamma_0, \gamma)$. Sticking to the notations of Proposition 1.2, we will prove the following

Proposition 1.3 *For each $\delta \in (0, \ell_0)$, there exists $r_2 \in (0, \min(R_1, r_1))$ such that, for each unit speed geodesic γ longer than δ and contained in $T_{r_2}(\gamma_0)$, a Fermi map can be constructed along γ with corresponding tube about γ containing the whole of $T(\gamma_0, \gamma)$ but its (γ_0, γ) -exceptional points.*

We call *family of Fermi maps near γ_0* the map which assigns, to each unit speed geodesic γ as stated and each (γ_0, γ) -accessible point $m \in T_{r_2}(\gamma_0)$, the image of m by the Fermi map along γ .

Proof. The idea is to use a suitable implicit function theorem argument along γ_0 . Since it is absent from the literature, we will present it carefully. Let us fix $\delta \in (0, \ell_0)$ and a unit speed geodesic $\sigma \in [0, \ell^*] \rightarrow \gamma^*(\sigma) \in T_{r_2}(\gamma_0)$, with $\ell^* \geq \delta$ and $r_2 \in (0, \min(R_1, r_1))$ to be chosen later. From Proposition 1.2, we know that γ^* is an embedding. We can thus construct a tube $T_\varrho(\gamma^*)$ about γ^* , for some radius $\varrho > 0$, as done for γ_0 in Section 1.1. We want $\rho_{\gamma^*} \leq r_2$ small enough compared to ϱ such that the tube $T_\varrho(\gamma^*)$ contains $T(\gamma_0, \gamma^*)$ but its exceptional points. Can we choose the radius r_2 such that this property holds for every such geodesic γ^* ?

First, we observe that the required property holds for γ^* if and only if it holds for the reversed geodesic γ_{rev}^* , given by: $\sigma \in [0, \ell^*] \rightarrow \gamma_{\text{rev}}^*(\sigma) = \gamma^*(\ell^* - \sigma)$. Therefore, applying Proposition 1.2 to γ^* , we may assume with no loss of generality that $\frac{d\gamma^{*n}}{d\sigma}$ is positive.

Next, we note that the geodesic γ^* is given by its Cauchy data $(p^*, u^*) = (\gamma^*(0), \frac{d\gamma^*}{d\sigma}(0)) \in UM$ and its length $\ell^* \in [\delta, L_0]$, while the generic point m^* of the tube $T_\varrho(\gamma^*)$ is determined by its Fermi map image $F_{\gamma^*}(m^*)$, namely by its height $\sigma^* = z_{\gamma^*}(m^*) \in [0, \ell^*]$ and by the vector $V^* = v_{\gamma^*}^\perp(m^*) \in (u^*)^\perp$ such that $|V^*| \leq \varrho$ and $E_{\gamma^*}(V^*, \sigma^*) = m^*$. Here, we have denoted by $E_{\gamma^*} : (u^*)^\perp \times (-\epsilon, \ell^* + \epsilon) \rightarrow M$ (resp. by $v_{\gamma^*}^\perp$) the analogue for γ^* of the map E_0 (resp. of the component v_0^\perp) defined for γ_0 at the beginning of Section 1.1.

The resulting point (p^*, u^*, V^*) , amalgam of the Cauchy data of γ^* with the Fermi component $V^* = v_{\gamma^*}^\perp(m^*) \in (u^*)^\perp$ of m^* , lies in the vector bundle $\ker T\pi \rightarrow UM$, kernel of the tangent map to the natural projection $\pi : UM \rightarrow M$. Sticking to the Fermi chart x along γ_0 , we use it to build a chart of $\ker T\pi$ near (p^*, u^*, V^*) by assigning to each neighboring point (p, u, V) the $(3n - 2)$ -tuple $(x^1, \dots, x^n, u_0^1, \dots, u_0^{n-1}, V_0^1, \dots, V_0^{n-1})$ with $x^i = x^i(p)$ and u_0^α, V_0^α defined as follows. Firstly, for each tangent vector $W \in T_p M$, let $\bar{W}_0 \in T_{p_0^\perp} M$, with $p_0^\perp = p_{\gamma_0}^\perp \equiv \gamma_0(x^n(p))$, denote its (backward) parallel transport⁶ along the geodesic ray $[\gamma_0, p]$, and $W_0 \in T_{\gamma_0(0)} M$, similarly from the latter now along γ_0 . We pause to record a lemma (the proof of which is left as an easy exercise):

Lemma 1.1 *If U is a unit tangent vector at $p \in T_{R_1}(\gamma_0)$ and \bar{U}_0 stands for its parallel transport to the point $\gamma_0(x^n(p))$ along the geodesic ray $[\gamma_0, p]$, the following estimate holds:*

$$\|U - \bar{U}_0\| \leq c_1 \Theta_0 r^2(p).$$

Applying this lemma, combined with Proposition 1.2 and the triangle inequality, to the vector $u^* \in T_{p^*} M$, and recalling that $\|\cdot\| \equiv |\cdot|$ along γ_0 , we infer the estimate:

$$(8) \quad |u_0^* - e_n| \leq k_1 r_2$$

with $k_1 = \frac{4}{\delta} + c_1 L_0 \Theta_0^2 + \left(c_1 \Theta_0 + c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{\delta} + c_1 L_0 \Theta_0^2 \right)^2 \right) r_1$. Here, we

used the positivity assumption made above on $(u^*)^n$. Taking $r_2 < \frac{1}{k_1}$, this estimate implies the positivity of $(u_0^*)^n$. Back to the definition of the chart of $\ker T\pi$ under elaboration, we take (p, u, V) close enough to (p^*, u^*, V^*) for u_0^n to be still positive, and we define the u_0^α 's and V_0^α 's by:

$$\sum_{\alpha=1}^{n-1} u_0^\alpha e_\alpha = \Pi_0 u_0, \quad \sum_{\alpha=1}^{n-1} V_0^\alpha e_\alpha = \Pi_0 V_0.$$

We recover the full parallel transported vectors u_0, V_0 , by setting $u_0^n = \sqrt{1 - \sum_{\alpha=1}^{n-1} (u_0^\alpha)^2}$ since $|u_0| = 1$ and $u_0^n > 0$, and $V_0^n = -\frac{1}{u_0^n} \sum_{\alpha=1}^{n-1} u_0^\alpha V_0^\alpha$

⁶henceforth, with respect to the Levi-Civita connection ∇ , unless otherwise specified

since $V_0 \perp u_0$. So $(x^i, u_0^\alpha, V_0^\alpha)$ is, indeed, a local chart of $\ker T\pi$. Although heavier, let us denote it rather by $(x^{*i}, u_0^{*\alpha}, V_0^{*\alpha})$ since we are now willing to move around the geodesic γ^* and the point $m^* \in T_\rho(\gamma^*)$, hence to let the point (p^*, u^*, V^*) itself vary in $\ker T\pi$ near $(p_0, u_0, V_0) = (\gamma_0(s_0), \frac{d\gamma_0}{ds}(s_0), 0)$ with $s_0 \in [0, \ell_0]$. Deferring the completion of the present proof, we pause to set up an appropriate implicit function theorem.

Implicit function theorem argument. In this paragraph, the requirement that the geodesic γ^* be longer than δ will be unnecessary, thus ignored provisionally. Given $s_0 \in [0, \ell_0]$ and $\sigma_0 \in [0, \ell_0 - s_0]$, let the point $(p^*, u^*, V^*) \in \ker T\pi$ be close to (p_0, u_0, V_0) and the real $\sigma^* \in \mathbb{R}^+$ be close to σ_0 ; let a further point m belong to $T_{r_2}(\gamma_0)$. Setting $\gamma^*(\sigma) = \exp_{p^*}(\sigma u^*)$ and $m^* = E_{\gamma^*}(V^*, \sigma^*)$, consider the map:

$$\Psi(p^*, u^*, V^*, \sigma^*, m) = x(m^*) - x(m) \in \mathbb{R}^n.$$

Using the chart $(x^{*i}, u_0^{*\alpha}, V_0^{*\alpha})$ for (p^*, u^*, V^*) and the chart x^i for m , let us denote the local expression of Ψ (resp. $x \circ E_{\gamma^*}$) by:

$$\Psi^i(x^{*j}, u_0^{*\alpha}, V_0^{*\alpha}, \sigma^*, x^j) = E^i(x^{*j}, u_0^{*\alpha}, V_0^{*\alpha}, \sigma^*) - x^i.$$

At the point given by⁷: $x^{*\alpha} = 0, x^{*n} = s_0; u_0^{*\alpha} = 0; V_0^{*\alpha} = 0; \sigma^* = \sigma_0; x^\alpha = 0, x^n = s_0 + \sigma_0$, we have:

$$\forall i \in \{1, \dots, n\}, \Psi^i((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0, (\vec{0}, s_0 + \sigma_0)) = 0, \text{ and}$$

$$\det \left(\frac{\partial \Psi^j}{\partial (V_0^{*\alpha}, \sigma^*)}((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0, (\vec{0}, s_0 + \sigma_0)) \right) \neq 0,$$

where $\vec{0}$ stands for the zero vector of \mathbb{R}^{n-1} . The latter equation holds since $\frac{\partial \Psi^j}{\partial (V_0^{*\alpha}, \sigma^*)} \equiv \frac{\partial E^j}{\partial (V_0^{*\alpha}, \sigma^*)}$ and $dE^j((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0) \equiv dx^j \circ dE_0(0, s_0 + \sigma_0)$ where $dE_0(0, s_0 + \sigma_0)$ is an *isomorphism* as seen in Section 1.1. We are thus in position to apply the Implicit Function Theorem [Lan02]. There exists a real $\epsilon > 0$ and a unique map $(x^{*j}, u_0^{*\alpha}, x^j) \rightarrow F^* = (\mathcal{V}_0^{*1}, \dots, \mathcal{V}_0^{*n-1}, \varsigma^*)$ such that, if:

$$(9) \quad \rho(x^*) \leq \epsilon, |x^{*n} - s_0| \leq \epsilon, |\Pi_0 u_0^*| \leq \epsilon, \rho(x) \leq \epsilon, |x^n - (s_0 + \sigma_0)| \leq \epsilon,$$

the identities: $\forall i \in \{1, \dots, n\}$,

$$\Psi^i(x^{*j}, u_0^{*\alpha}, \mathcal{V}_0^{*\alpha}(x^{*k}, u_0^{*\alpha}, x^k), \varsigma^*(x^{*k}, u_0^{*\alpha}, x^k), x^j) \equiv 0$$

⁷throughout with α ranging in $\{1, \dots, n-1\}$

are satisfied with $\sum_{\alpha=1}^{n-1} \left(\mathcal{V}_0^{*\alpha}(x^{*k}, u_0^{*\alpha}, x^k) \right)^2$ and $\left| \zeta^*(x^{*k}, u_0^{*\alpha}, x^k) - \sigma_0 \right|$ small.

By construction, these identities imply $m = m^*$; in other words, the map $x^j \rightarrow F^{*i}(x^{*j}, u_0^{*\alpha}, x^j)$ is nothing but the expression of the Fermi map F_{γ^*} along the geodesic $\gamma^*(\sigma) = \exp_{p^*}(\sigma u^*)$ read in the Fermi chart x along γ_0 . Finally, let us stress that the real $\epsilon > 0$ occurring in (9) may be chosen so small that it becomes independent of the couple of parameters (s_0, σ_0) , because the latter lies in a compact subset of \mathbb{R}^2 , namely in the triangle of the positive quadrant given by $s_0 + \sigma_0 \leq \ell_0$. Henceforth, we fix $\epsilon > 0$ so.

Completion of the proof of Proposition 1.3. Back to the case of our previous geodesic γ^* , supposed longer than δ and with positive axial component, we are now in position to choose the radius r_2 of the tube about γ_0 in which γ^* should lie. First of all, we fix a point $m \in T(\gamma_0, \gamma^*)$. So far, we have required $r_2 \in (0, \min(R_1, r_1, \frac{1}{k_1}))$. Redoing the preceding implicit function theorem argument now with $p^* = \gamma^*(0), s_0 = x^n(p^*), s_0 + \sigma_0 = x^n(m)$, the first and fourth inequalities of (9) prompt us to take $r_2 \leq \epsilon$. Besides, we must further shrink $r_2 > 0$ in order to keep γ^* nearly vertical so that the third inequality of (9) holds as well. From (8), we can do it by taking $r_2 \leq \frac{\epsilon}{k_1}$, as easily verified. Altogether, if the geodesic γ^* is longer than $\delta \in (0, \ell_0)$ with $\frac{d\gamma^{*n}}{d\sigma} > 0$ and if it is contained in the tube $T_{r_2}(\gamma_0)$ with $r_2 \in (0, \min(R_1, r_1, \frac{\epsilon}{k_1}))$, the triple:

$$\left(x^{*i} = x^{*i}(\gamma^*(0)), u_0^{*\alpha} = u_0^{*\alpha} \left(\frac{d\gamma^*}{d\sigma}(0) \right), x^i = x^i(m) \right)$$

satisfies the bounds (9). So we may consider its image by the local map F^* precendly constructed. In particular, it follows that the point m lies in a tube about the embedded geodesic γ^* if and only if its height $z_{\gamma^*}(m) = \zeta^*(x^{*i}, u_0^{*\alpha}, x^i)$ lies in the interval $[0, \ell^*]$. Since the point m was arbitrarily fixed in $T(\gamma_0, \gamma^*)$, we are done \square

1.5 Second fundamental form of a cylinder

If $n > 2$, sticking to the notations of Section 1.1, let us study the second fundamental form of a cylinder $C_r(\gamma_0)$ of small radius r about γ_0 .

Proposition 1.4 *Given $r \in (0, \min(1, R))$, a point $p \in C_r(\gamma_0)$ and a couple of vectors $(V, W) \in T_p C_r(\gamma_0) \times T_p C_r(\gamma_0)$, let us denote by $\Pi_p(V, W)$ the second fundamental form of the cylinder $C_r(\gamma_0)$ calculated at p on (V, W) . If we extend the vectors V, W and $N(p)$ as Fermi fields on $T_R(\gamma_0)$ and set*

$p^\perp = \gamma_0(z(p))$, the following asymptotic expansion holds:

$$\begin{aligned} \Pi_p(V, W) = & -\frac{1}{r} g(\Pi_0 V, \Pi_0 W)(p^\perp) + r \left(S(V, N(p), W, N(p))(p^\perp) \right. \\ & \left. - \frac{1}{3} S(\Pi_0 V, N(p), \Pi_0 W, N(p))(p^\perp) \right) + O(r^2), \end{aligned}$$

where, again, S stands for the sectional curvature tensor.

Proof. By definition [Gra04, p.33] [dC92, p.128], we have $\Pi_p(V, W) = g(-\nabla_V N, W)(p)$ and, here, one may allow the vectors V, W be arbitrary in $T_p M$ since N is vector field defined outside $C_r(\gamma_0)$. Covariant differentiation of the generalized Gauss lemma identity $g(N, \cdot) = d\mathfrak{t}$ on $\{\mathfrak{t} > 0\} \subset T_R(\gamma_0)$ yields:

$$(10) \quad \Pi_p(V, W) = -\nabla d\mathfrak{t}(V, W)(p).$$

More generally, for each couple of vector fields (A, B) , we find $\nabla d\mathfrak{t}(A, B) = g(A, \nabla_B N) = g(B, \nabla_A N)$ hence also, using Lie brackets:

$$(11) \quad 2 \nabla d\mathfrak{t}(A, B) = N.g(A, B) + g(A, [B, N]) + g(B, [A, N]),$$

since ∇ is torsionless. Taking a Fermi chart x along γ_0 such that $x(p) = (r, \underbrace{0, \dots, 0}_{n-2}, x^n(p))$, let us calculate $\nabla d\rho(r, 0, x^n)$ using (11) with A and B

equal to the $\frac{\partial}{\partial x^i}$'s. Note that $\nu(r, 0, x^n) = \frac{\partial}{\partial x^1}$ and $d\rho(r, 0, x^n) = dx^1$. From (1), we get $g_{1i}(r, 0, x^n) = \delta_{1i}$ and $N.g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^i}\right)(r, 0, x^n) = 0$. From (2), we find $\left[\frac{\partial}{\partial x^n}, \nu\right](r, 0, x^n) = 0$ and

$$\forall \alpha < n, \left[\frac{\partial}{\partial x^\alpha}, \nu\right](r, 0, x^n) = \frac{1}{r} \left(\frac{\partial}{\partial x^\alpha} - \delta_{1\alpha} \frac{\partial}{\partial x^1} \right);$$

in particular, $\left[\frac{\partial}{\partial x^1}, \nu\right](r, 0, x^n) = 0$. Besides, for $i, j \in \{2, \dots, n\}$, we can derive the local expressions of $N.g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(r, 0, x^n) = \frac{\partial g_{ij}}{\partial x^1}(r, 0, x^n)$ from the following Riemann type formulas extended to the Fermi setting [Spi79] [DG10, Lemma 2]:

$$g_{ab}(x^1, \underbrace{0, \dots, 0}_{n-2}, x^n) = \delta_{ab} - \frac{1}{3}(x^1)^2 R_{a1b1}(\underbrace{0, \dots, 0}_{n-1}, x^n) + O((x^1)^3),$$

with $a, b \in \{2, \dots, n-1\}$, and

$$g_{an}(x^1, 0, x^n) = -\frac{2}{3}(x^1)^2 R_{a1n1}(0, x^n) + O((x^1)^3),$$

$$g_{nn}(x^1, 0, x^n) = 1 - (x^1)^2 R_{n1n1}(0, x^n) + O((x^1)^3),$$

where x^1 stands for a small real parameter and R_{ijkl} for the components of the sectional curvature tensor. Doing so, we obtain the expression:

$$(12) \quad \begin{aligned} \nabla d\rho(r, 0, x^n) &= \sum_{a=2}^{n-1} \sum_{b=2}^{n-1} \left(\frac{1}{r} \delta_{ab} - \frac{2}{3} r R_{a1b1}(0, x^n) + O(r^2) \right) dx^a \otimes dx^b \\ &+ \sum_{a=2}^{n-1} \left(-r R_{a1n1}(0, x^n) + O(r^2) \right) (dx^a \otimes dx^n + dx^n \otimes dx^a) \\ &+ \left(-r R_{n1n1}(0, x^n) + O(r^2) \right) dx^n \otimes dx^n. \end{aligned}$$

The latter combined with (10) yields the proposition \square

Remark 1.1 For later use, we record here that, if $n = 2$, recalling (1), the expansion of the metric in the Fermi chart x becomes simply:

$$g(x^1, x^2) = dx^1 \otimes dx^1 + \left(1 - (x^1)^2 K(0, x^2) + O((x^1)^3) \right) dx^2 \otimes dx^2,$$

where K stands for the Gauss curvature of M . Accordingly, still from (11), the Hessian formula (12) becomes:

$$\nabla d\rho(r, x^2) = \left(-rK(0, x^2) + O(r^2) \right) dx^2 \otimes dx^2.$$

2 Further properties when the Jacobi operator is negative

From the properties established in the preceding section for a thin tube about the geodesic γ_0 , we will now derive stronger ones by assuming that the operator \mathcal{J}_{γ_0} is negative, as done in Theorem 0.1. Specifically, using the notations of Corollary 1.1 and setting $\bar{\kappa}_0 = \max_{s \in [0, \ell_0]} \kappa_0^{n-1}(s)$, our assumption means that $\bar{\kappa}_0 < 0$; henceforth, it is implicitly made.

Proposition 2.1 (the second fundamental forms stay definite) *One can find a small real $r_3 > 0$ such that, for each $p \in T_{r_3}(\gamma_0)$ with $r = \mathfrak{r}(p) \neq 0$, the second fundamental form of $C_r(\gamma_0)$ at the point p is negative definite.*

Proof. Let us take a Fermi chart x at the point p like the one used in the proof of Proposition 1.4 and write with it the expression of $\Pi_p(V, W)$ found

in that proposition, with $V = W = \sum_{i=2}^n V^i \frac{\partial}{\partial x^i} \in T_p C_r(\gamma_0)$. We find:

$$(13) \quad \begin{aligned} \Pi_p(V, V) &= -\frac{1}{2r} \sum_{a=2}^{n-1} (V^a)^2 + \frac{r}{2} R_{n1n1}(0, x^n) (V^n)^2 \\ &\quad - \frac{1}{4r} \left(\sum_{a=2}^{n-1} (V^a)^2 - 8r^2 \sum_{a=2}^{n-1} R_{a1n1}(0, x^n) V^a V^n - 2r^2 R_{n1n1}(0, x^n) (V^n)^2 \right) \\ &\quad - \frac{1}{4r} \sum_{a=2}^{n-1} \sum_{b=2}^{n-1} V^a V^b \left(\delta_{ab} - \frac{8}{3} r^2 R_{a1b1}(0, x^n) \right) + O(r^2) \end{aligned}$$

and the result readily follows from $R_{n1n1}(0, x^n) \leq \bar{\kappa}_0 < 0$, provided r is taken small enough \square

Proposition 2.2 (geodesics obey a Maximum Principle) *One can find a small real $r_4 > 0$ such that, for each geodesic path $t \in [0, 1] \rightarrow \gamma(t) \in T_{r_4}(\gamma_0)$, the following inequality holds:*

$$\max_{t \in [0, 1]} \mathfrak{r}(\gamma(t)) \leq \max(\mathfrak{r}(\gamma(0)), \mathfrak{r}(\gamma(1))).$$

Moreover, if $\mathfrak{r}(\gamma(\vartheta)) = \max(\mathfrak{r}(\gamma(0)), \mathfrak{r}(\gamma(1)))$ for some $\vartheta \in (0, 1)$, the path γ must be constant.

Proof. Anytime $t \in [0, 1] \rightarrow \gamma(t) \in T_R(\gamma_0)$ is a geodesic, at each $t \in [0, 1]$ such that $\mathfrak{r}(\gamma(t)) \neq 0$, we have:

$$\frac{d^2}{dt^2} (\mathfrak{r}(\gamma(t))) = \nabla d\mathfrak{r}(\gamma(t)) \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right).$$

If $n > 2$, combining (12) with (13) written with $V = \frac{d\gamma}{dt}$, we infer that the second derivative of the auxiliary real function $t \in [0, 1] \rightarrow \mathfrak{r}(\gamma(t))$ is *non negative* on $[0, 1]$ provided $\mathfrak{r}(\gamma(t)) \leq r_4 = r_3$. If $n = 2$, the same conclusion holds with r_4 small enough, due to Remark 1.1 read with $K(0, x^2) \leq \bar{\kappa}_0 < 0$. In any case, the Maximum Principle [PW67] implies the first part of the proposition. Moreover, it yields $\mathfrak{r} \circ \gamma \equiv \mathfrak{r}(\gamma(\vartheta)) =: r_\vartheta > 0$ hence $\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} C_{r_\vartheta}(\gamma_0)$ for each $t \in [0, 1]$. From (10) and Proposition 2.1 combined with $\frac{d^2}{dt^2} (\mathfrak{r}(\gamma(t))) \leq 0$, we infer that $\frac{d\gamma}{dt} \equiv 0$, so γ must indeed be constant \square

Before stating the next property, we require a lemma of independent interest⁸:

Lemma 2.1 *One can find a small real $r_5 > 0$ such that the quadratic forms inequality $g \geq d\mathfrak{r}_{\gamma_0}^2 + dz_{\gamma_0}^2$ holds at each point of $\{p \in T_{r_5}(\gamma_0), \mathfrak{r}(p) > 0\}$.*

⁸a reader who ignores the rest of the paper should understand it preceded by: "Let γ_0 be an embedded unit speed geodesic with negative Jacobi operator."

Proof. Take a point p as stated and a Fermi chart x along γ_0 such that $x(p) = (r, 0, \dots, 0, x^n)$. From Remark 1.1 read with $K(0, x^2) \leq \bar{\kappa}_0 < 0$, the lemma appears straightforward if $n = 2$. In higher dimension, from (1) and the expansion of $g_{ij}(x^1, 0, x^n)$ recalled above (see displayed formulas before (12)), we infer that, for each vector $V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i} \in T_p M$, the quadratic form $(g - d\mathfrak{r}_{\gamma_0}^2 - dz_{\gamma_0}^2)(p)$ applied to V can be expressed in the chart x , up to $O(r^3)$ terms, as the sum of two quadratic polynomials in V , namely $\sum_{a,b=2}^{n-1} \left(\frac{1}{2} \delta_{ab} - \frac{1}{3} r^2 R_{a1b1}(0, x^n) \right) V^a V^b$ and:

$$\sum_{a=2}^{n-1} \left(\frac{1}{2} V_a^2 - \frac{4}{3} r^2 R_{a1n1}(0, x^n) V^a V^n \right) - r^2 R_{n1n1}(0, x^n) (V^n)^2.$$

By taking $r > 0$ small enough, and using $R_{n1n1}(0, x^n) \leq \bar{\kappa}_0 < 0$ for the second polynomial, we can make each polynomial non negative \square

Proposition 2.3 (γ_0 is minimizing) *Take $r_5 > 0$ as in Lemma 2.1. The length of each piecewise C^1 path $t \in [0, 1] \rightarrow c(t) \in M$ ranging in $T_{r_5}(\gamma_0)$ with $z(c(0)) = 0$ and $z(c(1)) = \ell_0$, must be at least equal to ℓ_0 . Furthermore, if equality holds and $\mathfrak{r} \circ c(t) = 0$ for some $t \in [0, 1]$ then c , reparametrized by an arc length parameter suitably shifted to avoid jump⁹ on each subinterval of $[0, 1]$ in the interior of which c is C^1 and $\frac{dc}{dt} \neq 0$, coincides with γ_0 .*

Proof. Let c be a path as stated and x , a Fermi chart along γ_0 . From Lemma 2.1, the length of c satisfies: $\ell \geq \int_0^1 \sqrt{\left(\frac{d}{dt}(\rho \circ c) \right)^2 + \left(\frac{dc^n}{dt} \right)^2} dt$. Therefore, if $\int_0^1 \left| \frac{d}{dt}(\rho \circ c) \right| dt \neq 0$, we have $\ell > \int_0^1 \left| \frac{dc^n}{dt} \right| dt \geq \ell_0$ as asserted. Moreover, if $\ell = \ell_0$, we see that $\frac{d}{dt}(\rho \circ c)$ must vanish, hence also $(\rho \circ c)$ anytime it does at some $t \in [0, 1]$. In that case, the images of c and γ_0 coincide; so $\left| \frac{dc}{dt} \right| = \left\| \frac{dc}{dt} \right\| = \left| \frac{dc^n}{dt} \right|$ and $\int_0^1 \left| \frac{dc^n}{dt} \right| dt = \ell_0 = c^n(1) - c^n(0) = \int_0^1 \frac{dc^n}{dt} dt$. The latter equality implies that $\frac{dc^n}{dt} \geq 0$; so the path c , reparametrized by arc length as stated, must indeed coincide with γ_0 \square

Proposition 2.4 (long geodesics have a negative Jacobi operator) *Given $\delta > 0$, we can find $r_6 \in (0, \min(R_1, r_1)]$ such that, for each $r \in (0, r_6)$ and each unit speed geodesic $\sigma \in [0, \ell] \rightarrow \gamma(\sigma) \in T_r(\gamma_0)$ with length $\ell \geq \delta$, the Jacobi operator \mathcal{J}_γ is negative, or else $\max_{s \in [0, \ell]} \kappa_\gamma^{n-1}(s) < 0$.*

Proof. Let $k = k(r)$ be the affine function of r defined in Corollary 1.1 and r^+ be the positive root of the quadratic equation $rk(r) + \bar{\kappa}_0 = 0$; the proposition holds with $r_6 = \min(R_1, r_1, r^+)$ by Corollary 1.1 \square

⁹by taking the initial value of the parameter on a subinterval equal to (zero, of course, on the first subinterval and elsewhere to) to the final value of the parameter on the preceding subinterval

Proposition 2.5 (each geodesic is minimizing) *One can find a small real $r_7 > 0$ such that, for each unit speed geodesic $s \in [0, \ell] \rightarrow \gamma(s) \in M$ and each piecewise C^1 path $t \in [0, 1] \rightarrow c(t) \in M$, both ranging in $T_{r_7}(\gamma_0)$ with $c(0) = \gamma(0), c(1) = \gamma(\ell)$, the length of c must be at least equal to ℓ . Moreover, equality holds if and only if c , reparametrized by a suitable arclength parameter on each subinterval of $[0, 1]$ in the interior of which c is C^1 and $\frac{dc}{dt} \neq 0$, coincides with γ .*

Proof. Let γ be a geodesic of length ℓ as stated. The proposition is obvious if $\ell < i$. If $\ell \geq i$, which we suppose in the proof, we may use Propositions 1.2 and 1.3 read with $\delta = i$; the radii r_1 and r_2 are understood accordingly and we take $r_7 \leq r_2$. In this situation, we know that γ is an embedding and there exists a Fermi chart x_γ along γ whose domain $T_\eta(\gamma)$ contains $T(\gamma_0, \gamma)$ but the (γ_0, γ) -exceptional points.

Our next task is the main one, namely we must specify how the radius η of that tubular domain can be controlled by r_7 . By inspecting the proof of Proposition 1.3, we see (sticking to its notations, except for γ^* now written γ , so $m^* = \gamma(0), u^* = \frac{d\gamma}{ds}(0)$) that such a control amounts to a similar one on:

$$\|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\|^2 = \sum_{i=1}^n (\mathcal{V}_0^{*i}(x^*, \Pi_0 u_0^*, x))^2,$$

where $x^*, \Pi_0 u_0^*, x$ satisfy the bounds (9) now read with $\epsilon = r_7$ and where \mathcal{V}_0^{*n} has to be defined by:

$$\mathcal{V}_0^{*n} = -\frac{1}{u_0^{*n}} \sum_{\alpha=1}^{n-1} u_0^{*\alpha} \mathcal{V}_0^{*\alpha} \text{ with } u_0^{*n} = \pm \sqrt{1 - \sum_{\alpha=1}^{n-1} (u_0^{*\alpha})^2}.$$

Furthermore, as $r_7 \downarrow 0$, we know that $\sum_{\alpha=1}^{n-1} (\mathcal{V}_0^{*\alpha})^2$ tends to zero. All we require is thus a uniform positive lower bound on $|u_0^{*n}|$. Such a bound will follow from (5) and Lemma 1.1. Indeed, the former combined with Proposition 1.1 implies here: $\left| \frac{d\gamma^n}{ds} \right| \geq 1 - \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{i} + 2c_1 \Theta_0^2 (\ell_0 + i) \right)^2 \right) r_7^2$,

while the latter yields: $|u_0^{*n}| \geq \left| \frac{d\gamma^n}{ds} \right| - c_1 \Theta_0^2 r_7^2$, so we get:

$$|u_0^{*n}| \geq 1 - \left(2c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{i} + 2c_1 \Theta_0^2 (\ell_0 + i) \right)^2 \right) r_7^2.$$

Defining $\mathfrak{r}_1 > 0$ by, say: $\mathfrak{r}_1^2 \left(2c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left(\frac{4}{i} + 2c_1 \Theta_0^2 (\ell_0 + i) \right)^2 \right) = \frac{1}{2}$, and taking $r_7 \leq \mathfrak{r}_1$, we obtain $|u_0^{*n}| \geq \frac{1}{2}$. Now, it is clear that $\|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\|$ tends to zero as $r_7 \downarrow 0$. Here, among the arguments of \mathcal{V}_0^* , we are given the first one, since $x^* = x(\gamma(0))$; similarly for the second one, since $\Pi_0 u_0^*$ is defined out of $\frac{d\gamma}{ds}(0)$; the sole variable is the third one, since $x = x(m)$ with

$m \in T(\gamma_0, \gamma) \cap T_\eta(\gamma)$. Moreover, using the aforementioned Fermi chart x_γ , the identity $\rho(x_\gamma) = \|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\|$ holds. So $\rho(x_\gamma) \downarrow 0$ as $r_\gamma \downarrow 0$, which shows that the Implicit Function Theorem used in the proof of Proposition 1.3 allows us to let η go to zero as $r_\gamma \downarrow 0$.

Besides, Proposition 2.4 read with $\delta = i$ implies that, if we take $r_7 < r_6$, the Jacobi operator of γ is negative.

We conclude that there exists $r_7 > 0$ small enough such that, if γ ranges in $T_{r_7}(\gamma_0)$, the radius η of the tube about γ provided by Proposition 1.3 may be taken small enough such that Lemma 2.1 and Proposition 2.3 hold for the geodesic γ in $T_\eta(\gamma)$.

Now, we are in position to complete the proof of Proposition 2.5. Let c be a path as stated. By the definition of $T(\gamma_0, \gamma)$, the smallness of r_7 (hence of η) and the property of $T_\eta(\gamma)$ proved in Proposition 1.3, there exists a closed interval contained in $[0, 1]$ such that the restriction \bar{c} of c to this interval fulfills the assumption of Proposition 2.3 (read in $T_\eta(\gamma)$ instead of $T_{r_5}(\gamma_0)$). So we get the inequalities: $L = \text{length of } c \geq \text{length of } \bar{c} \geq \ell = \text{length of } \gamma$, which proves the first part of the proposition. Moreover, if $L = \ell$, the images of the paths c and \bar{c} must coincide, so \bar{c} shares with γ the same endpoints and the last part of Proposition 2.5 follows from that of Proposition 2.3 \square

Corollary 2.1 (each geodesic is uniquely determined by its endpoints)

Take $r_7 > 0$ as in Proposition 2.5. For each $(p, q) \in T_{r_7}(\gamma_0) \times T_{r_7}(\gamma_0)$, there exists at most one unit speed geodesic of $\gamma : [0, \ell] \rightarrow M$ entirely lying in $T_{r_7}(\gamma_0)$ with $\gamma(0) = p, \gamma(\ell) = q$.

Proof. By contradiction. If two distinct unit speed geodesics of M entirely lying in $T_{r_7}(\gamma_0)$ had the same endpoints, Proposition 2.5 would imply that the length of each geodesic be at least equal to the length of the other; so the geodesics would have equal length. Still by Proposition 2.5, the geodesics would thus coincide, which is absurd \square

3 Proof of Theorem 0.1

Reduction of the proof. We only have to prove the existence of a radius $r > 0$ such that each couple of points of the tube $T_r(\gamma_0)$, located as stated in Theorem 0.1, can be joined by a geodesic with interior lying in $\widehat{T_r(\gamma_0)}$. Indeed, suppose we did. Then, each such geodesic must be unique (by Corollary 2.1) and minimizing among piecewise C^1 paths sharing the same end points and lying in $T_r(\gamma_0)$ (by Proposition 2.5); so the proof is complete.

Strategy. Fixing $p \in T_r(\gamma_0)$, let us consider the subsets:

$$(14a) \quad \mathcal{Z}_p^+ = \{m \in T_r(\gamma_0), m \neq p, z(m) \geq z(p)\}$$

and, if $z(p) = 0$ or ℓ_0 , $z(m) \neq z(p)\}$,

$$(14b) \quad \mathcal{Z}_p^- = \{m \in T_r(\gamma_0), m \neq p, z(m) \leq z(p)\}$$

and, if $z(p) = 0$ or ℓ_0 , $z(m) \neq z(p)\}$.

Assuming $z(p) < \ell_0$, we will prove Theorem 0.1 for $q \in \mathcal{Z}_p^+$. Assuming $z(p) > 0$, we would prove it similarly for $q \in \mathcal{Z}_p^-$. Let us proceed to the proof itself. We distinguish two cases.

Case 1: $z(q) - z(p) < \frac{\mathfrak{c}}{2}$. For $\lambda \in [0, 1]$, set $p_\lambda^\perp = [\gamma_0, p](\lambda \mathfrak{r}(p))$ and $q_\lambda^\perp = [\gamma_0, q](\lambda \mathfrak{r}(q))$. Take $r < \frac{\mathfrak{c}}{2}$. Then, for each $\lambda \in [0, 1]$, the points p_λ^\perp and q_λ^\perp lie in the Riemannian ball $\{m \in M, d(m_0^\perp, m) < \varrho\}$ with center $m_0^\perp = \gamma_0\left(\frac{z(p) + z(q)}{2}\right)$ and radius $\varrho = \frac{\mathfrak{c}}{2} + r < \mathfrak{c}$. So there exists a unique minimizing geodesic $t \in [0, 1] \rightarrow c_\lambda(t) \in M$ such that $c_\lambda(0) = p_\lambda^\perp, c_\lambda(1) = q_\lambda^\perp$; here, for each $t \in [0, 1]$, the map $\lambda \in [0, 1] \rightarrow c_\lambda(t) \in M$ is smooth. We must prove that $c_1((0, 1)) \subset \widehat{T_r(\gamma_0)}$. To do so, let us argue by connectedness on the set:

$$\Lambda = \left\{ \lambda \in [0, 1], c_\lambda((0, 1)) \subset \widehat{T_r(\gamma_0)} \right\}.$$

By construction, Λ is non empty ($0 \in \Lambda$) and relatively open in $[0, 1]$, so we only have to prove that Λ is closed. Letting $(\lambda_i)_{i \in \mathbb{N}}$ be a sequence of Λ and $\lambda_\infty = \lim_{i \rightarrow \infty} \lambda_i \in [0, 1]$, it amounts to prove that $c_{\lambda_\infty}((0, 1)) \subset \widehat{T_r(\gamma_0)}$. By continuity, the geodesic c_{λ_∞} ranges in $T_r(\gamma_0)$. If $c_{\lambda_\infty}(\theta) \in C_r(\gamma_0)$ for some $\theta \in (0, 1)$, Proposition 2.2 implies that c_{λ_∞} is constant; so $p_{\lambda_\infty}^\perp = q_{\lambda_\infty}^\perp$. But the latter yields $p = q$, contradicting the assumption $q \in \mathcal{Z}_p^+$.

We are left with ruling out the following property:

$$(15) \quad \exists \theta \in (0, 1), z(c_{\lambda_\infty}(\theta)) = 0 \text{ or } \ell_0.$$

To do so, given $\delta > 0$, we distinguish two subcases as stated in Theorem 0.1.

Subcase 1: $n = 2$. If (15) held, the vector $\frac{dc_{\lambda_\infty}}{dt}(\theta)$ would necessarily belong to $\ker dz \setminus \{0\}$. But then, the geodesic $t \mapsto c_{\lambda_\infty}(t)$ would stay for all $t \in [0, 1]$ in the end of the tube given by the equation $z = z(c_{\lambda_\infty}(\theta))$ because, when $n = 2$, the latter is *totally geodesic*. We reach a contradiction, since we have assumed that $z(p) < \ell_0$ and, if $z(p) = 0, z(q) \neq 0$.

Subcase 2: $n > 2$ and either $|z(p) - z(q)| \geq \varsigma$ or $\varsigma \leq z(p) \leq z(q) \leq \ell_0 - \varsigma$. If $|z(p) - z(q)| \geq \varsigma$, the length ℓ_{λ_∞} of the geodesic c_{λ_∞} must be bounded below by ς due to Lemma 2.1. It follows that $\frac{dc_{\lambda_\infty}^n}{dt} > 0$ if $r > 0$ is taken small enough, due to Proposition 1.2 read with $\delta = \varsigma$. So, in that case, the property (15) cannot hold. If instead $\varsigma \leq z(p) \leq z(q) \leq \ell_0 - \varsigma$, with $|z(p) - z(q)| < \varsigma$, the latter inequality yields $\ell_{\lambda_\infty} \leq \varsigma + 2r$, while the former pinching combined with Lemma 2.1 yields $\ell_{\lambda_\infty} \geq 2\varsigma$ if (15) holds. In that

case, we get the lower bound $r \geq \varsigma$ which is absurd, provided $r < \varsigma$. In either case, we conclude that (15) cannot occur for $r > 0$ small enough.

Having proved that $\lambda_\infty \in \Lambda$, we conclude that Λ is closed hence equal to $[0, 1]$. In particular, $1 \in \Lambda$ so Case 1 is settled. \square

Case 2: $z(q) - z(p) \geq \frac{\mathfrak{c}}{2}$. Here, reading the constant r_1 from Proposition 1.2 with $\delta = \frac{\mathfrak{c}}{2}$, we take $r > 0$ small as done in Proposition 2.4. Furthermore, we consider the subset of the interval $[z(p), \ell_0]$ defined by:

$$\mathfrak{Z}_p^+ = \left\{ \mathfrak{z} \in [z(p), \ell_0], \forall m \in \mathcal{Z}_p^+, z(m) = \mathfrak{z} \implies T_r(\gamma_0) \text{ is strongly convex for } (p, m) \right\}.$$

By construction, if $\mathfrak{z} \in \mathfrak{Z}_p^+$, the whole interval $[z(p), \mathfrak{z}]$ must lie in \mathfrak{Z}_p^+ and, by Case 1, we know that \mathfrak{Z}_p^+ contains the interval $[z(p), z(p) + \frac{\mathfrak{c}}{2}]$. In the next two lemmas, we prove that \mathfrak{Z}_p^+ is both closed and relatively open in $[z(p), \ell_0]$. Granted it is, by connectedness, it must coincide with $[z(p), \ell_0]$ hence Theorem 0.1 is established when $z(p) < \ell_0$ and $q \in \mathcal{Z}_p^+$. The proof when $z(p) > 0$ and $q \in \mathcal{Z}_p^-$ is similar \square

Lemma 3.1 *The subset \mathfrak{Z}_p^+ is closed.*

Lemma 3.2 *The subset \mathfrak{Z}_p^+ is relatively open in $[z(p), \ell_0]$.*

Proof of Lemma 3.1. Let $(z_i)_{i \in \mathbb{N}}$ be a sequence of \mathfrak{Z}_p^+ ; set $\mathfrak{z} = \lim_{i \rightarrow \infty} z_i \in [z(p), \ell_0]$. We must prove that $\mathfrak{z} \in \mathfrak{Z}_p^+$, so we may assume with no loss of generality that $\mathfrak{z} \geq z(p) + \frac{\mathfrak{c}}{2}$. Fix $m \in \mathcal{Z}_p^+$ satisfying $z(m) = \mathfrak{z}$ and let $(m_i)_{i \in \mathbb{N}}$ be a sequence of \mathcal{Z}_p^+ such that: $\forall i \in \mathbb{N}, z(m_i) = z_i$ and $\lim_{i \rightarrow \infty} m_i = m$. For each $i \in \mathbb{N}$, set $t \in [0, 1] \rightarrow c_i(t) \in M$ for the unique minimizing geodesic such that $c_i(0) = p, c_i(1) = m_i$ and $c_i((0, 1)) \subset \widehat{T_r(\gamma_0)}$. By Proposition 1.1, the sequence $\left(\frac{dc_i}{dt}(0) \right)_{i \in \mathbb{N}}$ is bounded in $T_p M$, it thus converges toward a vector $V \in T_p M$. By continuity of the map $\exp_p : T_p M \rightarrow M$, the geodesic $t \in [0, 1] \rightarrow \exp_p(tV) \in M$ (let us denote it by c) satisfies $c(0) = p, c(1) = m$ and $c([0, 1]) \subset T_r(\gamma_0)$. For each $t \in (0, 1)$, Proposition 1.2 implies that $z(c(t)) \in (z(p), z(m))$ while, taking $r \leq r_4$, we know that $\mathfrak{r}(c(t)) < r$ by

Proposition 2.2; so the inclusion $c((0, 1)) \subset \widehat{T_r(\gamma_0)}$ must hold. Finally, by Proposition 2.5 and Corollary 2.1, the geodesic c must be minimizing and unique in $T_r(\gamma_0)$. In other words, we have proved that $T_r(\gamma_0)$ is strongly convex for (p, m) . Since the point m is arbitrary, we conclude that $\mathfrak{z} \in \mathfrak{Z}_p^+$ as desired \square

Proof of Lemma 3.2. Pick $\mathfrak{z} \in \mathfrak{Z}_p^+$ and $m \in \mathcal{Z}_p^+$ with $z(m) = \mathfrak{z}$. We may take $\mathfrak{z} \in \left[z(p) + \frac{\mathfrak{c}}{2}, \ell_0 \right)$ without loss of generality, due to Lemma 3.1. Let

$t \in [0, 1] \rightarrow c_m(t) \in M$ be the geodesic such that: $c_m(0) = p, c_m(1) = m$ and $c_m((0, 1)) \subset \widehat{T_r(\gamma_0)}$. By Proposition 2.4, the Jacobi operator of c_m is negative. Therefore the tangent map $d(\exp_p)\left(\frac{dc_m}{dt}(0)\right) : T_pM \rightarrow T_mM$ is invertible [Aub98, pp.17–18] [dC92, p.117, p.149] [Mil63, p.98, p.100]. The Inverse Function Theorem [Lan02] yields a real $\epsilon_m > 0$ such that each point m' lying in the Riemannian ball $B(m, \epsilon_m)$ can be joined to the point p by a unique geodesic $t \in [0, 1] \rightarrow c_{m'}(t) = \exp_p(tV') \in M$ with $V' \in T_pM$ close to $V_m = \frac{dc_m}{dt}(0)$. Possibly shrinking $\epsilon_m > 0$, we take it such that $z(p) + \frac{\mathfrak{c}}{4} \leq z < \ell_0$ on $\overline{B(m, \epsilon_m)}$. Since the level set $T_r(\gamma_0) \cap \{z = \mathfrak{z}\}$ is compact, it can be covered by the union of finitely many balls $B_i = B(m_i, \epsilon_i), i \in \{1, \dots, N\}$, each constructed like the ball $B(m, \epsilon_m)$. There exists $\theta > 0$ such that the level set $T_r(\gamma_0) \cap \{z = \mathfrak{z} + \theta\}$ remains covered by $\bigcup_{i=1}^N B_i$.

Claim. $\mathfrak{z} + \theta$ belongs to the subset \mathfrak{Z}_p^+ .

The claim, provisionally taken for granted, implies that $[z(p), \mathfrak{z} + \theta] \subset \mathfrak{Z}_p^+$, so Lemma 3.2, indeed, holds \square

Proof of the Claim. Pick $m' \in \mathfrak{Z}_p^+$ with $z(m') = \mathfrak{z} + \theta$. There exists $i \in \{1, \dots, N\}$ such that $m' \in B_i$. So $m' = \exp_p(V')$ for a unique vector $V' \in T_pM$ close to $V_i = \frac{dc_{m_i}}{dt}(0)$. Moreover, there exists a unique geodesic path $\lambda \in [0, 1] \rightarrow \mathbf{m}(\lambda) \in M$ ranging in B_i such that $\mathbf{m}(0) = m_i, \mathbf{m}(1) = m'$. Let $\lambda \in [0, 1] \rightarrow V_\lambda \in T_pM$ be the corresponding path, derived (like V') from the Inverse Function Theorem as done above, such that $\exp_p(V_\lambda) \equiv \mathbf{m}(\lambda)$. Set $t \in [0, 1] \rightarrow \gamma_\lambda(t) \in M$ for the geodesic path given by $\gamma_\lambda(t) = \exp_p(tV_\lambda)$. From the pinching $z(p) + \frac{\mathfrak{c}}{4} \leq z(\mathbf{m}(\lambda)) < \ell_0$ combined with Proposition 2.2,

we know that $\mathbf{m}((0, 1)) \subset \widehat{T_r(\gamma_0)}$. Let us argue by connectedness on the subset of the interval $[0, 1]$ given by:

$$L = \left\{ \lambda \in [0, 1], \gamma_\lambda((0, 1)) \subset \widehat{T_r(\gamma_0)} \right\},$$

which is non empty ($0 \in L$). The closedness of L can readily be established, arguing as we did for that of \mathfrak{Z}_p^+ . Let us focus on proving that L is relatively open in $[0, 1]$. If $\lambda \in L$, the continuity of \exp_p implies the existence of $\mu > 0$ such that $\gamma_{\lambda'}([0, 1]) \subset T_{2r}(\gamma_0)$ for each $\lambda' \in (\lambda - \mu, \lambda + \mu) \cap [0, 1]$. By Lemma 2.1, taking $2r \leq r_5$, we know that the length of the geodesic $\gamma_{\lambda'}$ is at least equal to $\frac{\mathfrak{c}}{4}$. By Proposition 1.2 read in $T_{2r}(\gamma_0)$ with $\delta = \frac{\mathfrak{c}}{4}$,

we can take $r > 0$ small enough such that $\frac{d\gamma_{\lambda'}}{dt} > 0$, hence $z(\gamma_{\lambda'}((0, 1])) \subset (z(p), \ell_0)$. Furthermore, taking $2r \leq r_4$ and applying Proposition 2.2, we get $\mathfrak{r}(\gamma_{\lambda'}(t)) < r$ for $t \in (0, 1)$. It follows that $\lambda' \in L$, in other words, L is relatively open in $[0, 1]$. By connectedness, we get: $L = [0, 1]$. In particular,

$1 \in L$, from what we readily infer that $m' \in \mathcal{Z}_p^+$. Since m' is arbitrary, we conclude: $\mathfrak{z} + \theta \in \mathfrak{Z}_p^+$, as claimed \square

4 Proof of Corollary 0.1

The assumption made in Theorem 0.1 on the geodesic γ_0 is an *open* condition. Given a small real $\varsigma > 0$, we can thus find $r > 0$ such that Theorem 0.1 holds for the geodesic $s \in [-r, \ell_0 + r] \rightarrow \gamma_r(s) \in M$ defined as the extension of the geodesic γ_0 to the interval $[-r, \ell_0 + r]$. There still exists a Fermi map about the extended geodesic γ_r ; let us stick to our preceding notations for this map. It is important to note the inclusion:

$$(16) \quad N_r(\gamma_0) \subset T_r(\gamma_r)$$

which follows from those of $\overline{B(\gamma_0(0), r)}$ and $\overline{B(\gamma_0(\ell_0), r)}$ in $T_r(\gamma_r)$ combined with the identity: $N_r(\gamma_0) \equiv T_r(\gamma_0) \cup \overline{B(\gamma_0(0), r)} \cup \overline{B(\gamma_0(\ell_0), r)}$. Given a couple of points (p, q) in $N_r(\gamma_0)$, say with $z(p) \leq z(q)$, we must prove that $N_r(\gamma_0)$ is strongly convex for (p, q) . To do so, it suffices to construct a

geodesic path from p to q ranging in $\widehat{N_r(\gamma_0)}$. Indeed, by (16) combined with Proposition 2.5 and Corollary 2.1 applied in $T_r(\gamma_r)$, such a geodesic path will necessarily be minimizing and unique in $N_r(\gamma_0)$. From Theorem 0.1 applied in $T_r(\gamma_0) \subset N_r(\gamma_0)$, we only have to treat the following two cases.

Case 1: $z(q) - z(p) \geq \varsigma$ and, either $z(p) < 0$ or $z(q) > \ell_0$. By Theorem 0.1, the tube $T_r(\gamma_r)$ is strongly convex for (p, q) . Let $t \in [0, 1] \rightarrow \gamma(t) \in M$ denote the geodesic from $\gamma(0) = p$ to $\gamma(1) = q$ such that $\gamma((0, 1)) \subset \widehat{T_r(\gamma_r)}$.

We must prove that $\gamma((0, 1)) \subset \widehat{N_r(\gamma_0)}$. By Proposition 1.2, we know that $\frac{d(z \circ \gamma)}{dt} > 0$ while, by Proposition 2.2, we have $\mathfrak{r} \circ \gamma < r$ on $(0, 1)$. We may assume with no loss of generality the existence of $T \in (0, 1)$ such that, either $z(\gamma(T)) = 0$ or $z(\gamma(T)) = \ell_0$. If the former (resp. latter) occurs, the restriction of γ to the subinterval $[0, T]$ (resp. $[T, 1]$) is minimizing in $T_r(\gamma_r) \cap \{-r \leq z \leq 0\}$ (resp. $T_r(\gamma_r) \cap \{\ell_0 \leq z \leq \ell_0 + r\}$) among piecewise C^1 paths joining p to $\gamma(T)$ (resp. $\gamma(T)$ to q). Besides, the ball $\overline{B(\gamma_0(0), r)}$ (resp. $\overline{B(\gamma_0(\ell_0), r)}$) being strongly convex, there exists a unique minimizing geodesic $\tau \in [0, 1] \rightarrow c(\tau) \in M$ such that $c(0) = p, c(1) = \gamma(T), c((0, 1)) \subset \overset{\circ}{B}(\gamma_0(0), r)$ (resp. $c(0) = \gamma(T), c(1) = q$ and $c([0, 1]) \subset \overset{\circ}{B}(\gamma_0(\ell_0), r)$). By uniqueness and due to (16), these geodesics must coincide: $c(\tau) \equiv \gamma(\tau T)$ (resp. $c(\tau) \equiv \gamma(\tau + (1 - \tau)T)$). In particular, we do have $\gamma((0, T]) \subset \overset{\circ}{B}(\gamma_0(0), r)$ (resp. $\gamma([T, 1]) \subset \overset{\circ}{B}(\gamma_0(\ell_0), r)$). Case 1 is settled.

Case 2: $z(q) - z(p) < \varsigma$ and, either $z(p) < \varsigma$ or $z(q) > \ell_0 - \varsigma$. Here, we may assume that the points p and q lie in the closure of a strongly convex

ball B and argue as in Case 1 of the proof of Theorem 0.1, with $T_r(\gamma_0)$ now replaced by $N_r(\gamma_0)$. Doing so, the present proof is reduced to ruling out the analogue of (15), namely the property:

$$\exists \theta \in (0, 1), c_{\lambda_\infty}(\theta) \in [\partial B(\gamma_0(0), r) \cap \{z < 0\}] \cup [\partial B(\gamma_0(\ell_0), r) \cap \{z > \ell_0\}].$$

It can be done by observing that the geodesic $t \in [0, 1] \rightarrow c_{\lambda_\infty}(t) \in M$ is *minimizing* from $p_{\lambda_\infty}^\perp$ to $q_{\lambda_\infty}^\perp$ and by relying on the inclusion (16) combined with the strong convexity of the balls $\overline{B(\gamma_0(0), r)}$ and $\overline{B(\gamma_0(\ell_0), r)}$; we leave it as an exercise.

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