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# Perfect simulation for the Feynman-Kac law on the path space

Christophe Andrieu\*, Nicolas Chopin †, Arnaud Doucet‡, Sylvain Rubenthaler§

1st October 2012

## Abstract

This paper describes an algorithm of interest. This is a preliminary version and we intend on writing a better description of it and getting bounds for its complexity.

## 1 Introduction

We are given a transition kernel  $M$  (on a space  $E$ ),  $M_1$  a probability measure on  $E$  and potentials  $(G_k)_{k \geq 1}$  ( $G_k : E \rightarrow \mathbb{R}_+$ ). We want to draw samples according to the law (on paths of length  $P$ )

$$\pi(f) = \frac{\mathbb{E}(f(X_1, \dots, X_P) \prod_{i=1}^{P-1} G_i(X_i))}{\mathbb{E}(\prod_{i=1}^{P-1} G_i(X_i))}$$

where  $(X_k)$  is Markov with initial law  $M_1$  and transition  $M$ . For all  $n \in \mathbb{N}$ , we note  $[n] = \{1, \dots, n\}$ .

## 2 Densities of branching processes

### 2.1 Description of a branching system

We start with  $N_1$  particles (i.i.d. with law  $M_1$ ,  $N_1$  is a fixed number). If we have  $N_i$  particles at time  $n$ , the system evolves in the following manner:

- The number of children of  $X_n^i$  (the  $i$ -th particle at time  $n$ ) is a random variable  $A_{n+1}^i$  with law  $f_{n+1}$  such that :  $\mathbb{P}(A_{n+1}^i = j) = f_{n+1}(G_n(X_n^i), j)$  (here,  $f_n$  is a law with a parameter  $G_n(X_n^i)$ , we will define this law later). The variables  $A_{n+1}^i$  ( $1 \leq i \leq N_n$ ) are independent. We then have  $N_{n+1} = \sum_{i=1}^{N_n} A_{n+1}^i$
- We draw  $\sigma_{n+1}$  uniformly in  $\mathcal{S}_{N_{n+1}}$  (the  $N_{n+1}$ -th symmetric group).
- We set  $\forall j \in [N_{n+1}]$ ,  $B_{n+1}^j = \{A_{n+1}^1 + \dots + A_{n+1}^{j-1}, \dots, A_{n+1}^1 + \dots + A_{n+1}^{j-1} + A_{n+1}^j\}$ . If  $i \in \sigma_{n+1}(B_{n+1}^j)$ , we draw  $X_{n+1}^i \sim M(X_n^j, \cdot)$ .

Such a system has a density on the space

$$\{(n_2, \dots, n_p, x_n^i, A_n^i, \sigma_n) : n_2, \dots, n_p \in \mathbb{N},$$

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$$x_n^i \in E(1 \leq n \leq P, 1 \leq i \leq n_n), A_n^i \in \mathbb{N}(2 \leq n \leq P, 1 \leq i \leq n_n), \sigma_n \in \mathcal{S}_{N_n}(2 \leq n \leq P).$$

This density is equal to :

$$\begin{aligned} q_0(N_2, \dots, N_P, (A_n^i)_{2 \leq n \leq P, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}) \\ = \prod_{i=1}^{N_1} M_1(x_1^i) \prod_{n=2}^P \prod_{i=1}^{N_{n-1}} f_n(G_{n-1}(x_{n-1}^i), A_n^i) \frac{1}{N_n!} \prod_{j \in \sigma_n(B_n^i)} M(x_{n-1}^i, x_n^j). \end{aligned}$$

The random permutations  $\sigma_N$  ease the writing of the formulas but have no deep signification.

## 2.2 Proposal density

We take the above branching system and we draw a path by drawing a number  $i$  uniformly in  $\{1, \dots, N_P\}$  and taking the path of the ancestors of  $X_P^i$ . The branching system plus this trajectory live in the following space

$$\begin{aligned} \{(n_2, \dots, n_P, x_n^i, A_n^i, \sigma_n, b_i) : n_2, \dots, n_P \in \mathbb{N}, \\ x_n^i \in E(1 \leq n \leq P, 1 \leq i \leq n_n), A_n^i \in \mathbb{N}(2 \leq n \leq P, 1 \leq i \leq n_i), \\ \sigma_n \in \mathcal{S}_{N_n}(2 \leq n \leq P), b_i \in [N_i](1 \leq i \leq n)\}, \quad (2.1) \end{aligned}$$

and have the following density :

$$\begin{aligned} q(N_2, \dots, N_P, (A_n^i)_{2 \leq n \leq P, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}, (b_k)_{1 \leq k \leq P}) \\ = \frac{1}{N_P} q_0(N_2, \dots, N_P, (A_n^i)_{2 \leq n \leq P, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}). \end{aligned}$$

## 2.3 Target law

We draw a trajectory  $(y_1, \dots, y_P)$  with the law  $\pi$  then a branching system conditioned on containing the trajectory  $(y_1, \dots, y_P)$ . The order of operations is as followed

- Draw  $(y_1, \dots, y_P)$  with law  $\pi(\cdot)$ .
- We draw  $b_1$  uniformly in  $[N_1]$ , we set  $x_1^{b_1} = y_1$ . We draw  $(x_1^i)_{1 \leq i \leq N_1, i \neq b_1}$  i.i.d. variables of law  $M_1$ .
- If we have the  $(n-1)$ -th generation, we draw  $A_n^{b_{n-1}}$  with law  $f(G_{n-1}(x_{n-1}^{b_{n-1}}), \cdot)$  conditioned to be in  $\mathbb{N}^*$  (we call this law  $\hat{f}(G_{n-1}(x_{n-1}^{b_{n-1}}), \cdot)$ ). For  $i \in N_{n-1}$ ,  $i \neq b_{n-1}$ , we draw  $A_n^i \sim f_n(G_{n-1}(x_{n-1}^i), \cdot)$ . We set  $N_n = \sum_{i=1}^{N_{n-1}} A_n^i$ . We draw  $\sigma_n$  uniformly in  $\mathcal{S}_{N_n}$ . We set  $b_n = \sigma_n(A_n^1 + \dots + A_n^{b_{n-1}-1} + 1)$ ,  $x_n^{b_n} = y_n$ . For  $j \in [N_n]$ , if  $j \neq b_n$  and  $j \in \sigma_n(B_n^i)$  ( $B_n^i = \{A_n^1 + \dots + A_n^{i-1} + 1, \dots, A_n^i + \dots + A_n^j\}$ ), we draw  $x_n^j \sim M(x_{n-1}^i, \cdot)$ .

We get a variable in the following space

$$\begin{aligned} \{(n_2, \dots, n_P, x_n^i, A_n^i, \sigma_n, b_i) : n_2, \dots, n_P \in \mathbb{N}^*, x_n^i \in E(1 \leq n \leq P, 1 \leq i \leq n_n), \\ A_n^i \in \mathbb{N}(1 \leq n \leq P, 1 \leq i \leq n_n), \sigma_n \in \mathcal{S}_{N_n}(2 \leq n \leq P), b_i \in [N_i](1 \leq i \leq n)\}, \end{aligned}$$

with the following density:

$$\begin{aligned} \hat{\pi}(N_2, \dots, N_P, (A_n^i)_{2 \leq n \leq P, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}, (b_k)_{1 \leq k \leq P}) \\ = \pi(x_1^{b_1}, \dots, x_P^{b_P}) \frac{1}{N_1} \prod_{1 \leq i \leq N_1, i \neq b_1} M_1(x_1^i) \end{aligned}$$

$$\prod_{n=2}^P \left( \widehat{f}_n(G_{n-1}(x_{n-1}^{b_{n-1}}), A_n^{b_{n-1}}) \prod_{1 \leq i \leq N_{n-1}, i \neq b_{n-1}} f_n(G_{n-1}(x_{n-1}^i), A_n^i) \right. \\ \left. \times \frac{1}{N_n!} \prod_{1 \leq i \leq N_{n-1}} \prod_{j \in \sigma_n(B_n^i), j \neq b_n} M(x_{n-1}^i, x_n^j) \right). \quad (2.2)$$

Notice that:  $(\forall z, k) \widehat{f}_n(g, k) = \frac{f_n(g, k)}{1 - f_n(g, 0)}$  ( $x_{n-1}^{b_{n-1}}$  is conditioned on having at least one children).

## 2.4 Ratio of the densities

We write the ratio  $\widehat{\pi}/q$  and we get:

$$\frac{\widehat{\pi}(N_2, \dots, N_P, (A_n^i)_{1 \leq n \leq P-1, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}, (b_k)_{1 \leq k \leq P})}{q(N_2, \dots, N_P, (A_n^i)_{1 \leq n \leq P-1, 1 \leq i \leq N_n}, (x_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n}, (\sigma_n)_{2 \leq n \leq P}, (b_k)_{1 \leq k \leq P})} \\ = \pi(x_1^{b_1}, \dots, x_P^{b_P}) \times \frac{N_P}{N_1} \times \frac{1}{M_1(x_1^{b_1}) \prod_{n=2}^P M(x_{n-1}^{b_{n-1}}, x_n^{b_n})} \times \prod_{n=2}^P \frac{\widehat{f}_n(G_{n-1}(x_{n-1}^{b_{n-1}}), A_n^{b_{n-1}})}{f_n(G_{n-1}(x_{n-1}^{b_{n-1}}), A_n^{b_{n-1}})}.$$

Let us take  $f_n$  such that for all  $g, i$  ( $i \neq 0$ ),  $\frac{\widehat{f}_n(g, i)}{f_n(g, i)} = \frac{\beta_n}{g}$  for some constant  $\beta_n$ . This means that  $1 - f_n(g, 0) = \frac{g}{\beta_n}$ . We then get:

$$\frac{\widehat{\pi}(\dots)}{q(\dots)} = \frac{N_P \prod_{i=2}^P \beta_i}{N_1 Z},$$

with  $Z = \mathbb{E}(\prod_{n=1}^{P-1} G_n(X_n))$  ( $(X_n)_{n \geq 1}$  is a Markov chain with initial law  $M_1$  and kernel transition  $M$ ).

## 3 Perfect simulation algorithm

### 3.1 Stability of the branching process

We want the branchin process to be stable. So we need that

$$\frac{1}{N_{n-1}} \sum_{i=1}^{N_{n-1} + \infty} \sum_{j=1}^{N_{n-1} + \infty} j f_n(G_{n-1}(x_{n-1}^i), j) \text{ be of order } 1 \text{ } (\forall n). \quad (3.1)$$

Let us take:  $\beta_n \geq \|G_n\|_\infty$  ( $\forall n$ ), and (for some  $k_n$ ),  $f_n(g, 0) = 1 - \frac{g}{\beta_n}$ ,  $f_n(g, i) = \frac{g}{k_n \beta_n}$  pour  $1 \leq i \leq k_n$ . We then get  $\sum_{i=1}^{k_n} i \times f_n(g, i) = \frac{(k_n+1)g}{2\beta_n}$ . So it is sensible to fix  $k_n$  such that

$$\beta_n = \frac{k_n + 1}{2} \times \frac{1}{N} \sum_{i=1}^N G_{n-1}(\bar{x}_{n-1}^i) \quad (3.2)$$

where  $(\bar{x}_{n-1}^i)$  is a sequential Monte-Carlo system with  $N$  particles, this has to be computed beforehand. Simulations show that this procedure indeed gives you stable branching processes.

### 3.2 Markovian transition

We know want to use a backward coupling algorithm (as in [FT98, PW96]). The integer  $N_1$  is fixed. We take  $(z_1, \dots, z_P) \in E^P$ .

- We draw  $N_2, \dots, N_P, (X_n^i)_{1 \leq n \leq P, 1 \leq i \leq N_n, i \neq B_n}, (A_n^i)_{1 \leq n \leq P, 1 \leq i \leq n_n}, (S_n \in \mathcal{S}_{N_n})_{2 \leq n \leq P}, (B_k)_{1 \leq k \leq P}$  with the density

$$\frac{\widehat{\pi}(\dots, z_1, \dots, z_P, \dots)}{\pi(z_1, \dots, z_P)} \quad (3.3)$$

( $z_1, \dots, z_P$  in place of  $x_1^{b_1}, \dots, x_P^{b_P}$  in equation (2.2)). This amounts to drawing a genealogy conditioned to contain  $(z_1, \dots, z_P)$ . Let us set  $\forall n \in \{1, \dots, P\}, X_n^{B_n} = z_n$ . Let  $\mathcal{X}$  be the variable containing all the  $N_n, X_n^i, A_n^i, S_n, B_n$ .

- We draw  $\bar{N}_2, \dots, \bar{N}_P (\bar{X}_n^i)_{1 \leq n \leq P, 1 \leq i \leq \bar{N}_n}, (\bar{A}_n^i)_{1 \leq n \leq P, 1 \leq i \leq \bar{N}_n}, (\bar{S}_n \in \mathcal{S}_{N_n})_{2 \leq n \leq P}, (\bar{B}_k)_{1 \leq k \leq P}$  with density  $q(\cdot)$ . We denote by  $\bar{\mathcal{X}}$  the corresponding variable.
- With probability  $\inf\left(1, \frac{\widehat{\pi}(\bar{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\bar{\mathcal{X}})}\right)$ , we set  $(\bar{Z}_1, \dots, \bar{Z}_P) = (\bar{X}_1^{B_1}, \dots, \bar{X}_P^{B_P})$ , and with the complementary probability, we set  $(\bar{Z}_1, \dots, \bar{Z}_P) = (z_1, \dots, z_P)$ .

The transformation of  $(z_1, \dots, z_P)$  into  $(\bar{Z}_1, \dots, \bar{Z}_P)$  is a Metropolis Markov kernel (on  $E^P$ ) for which  $\pi$  is invariant (much in the spirit of [ADH10]). Recall that

$$\frac{\widehat{\pi}(\bar{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\bar{\mathcal{X}})} = \frac{\bar{N}_P}{N_P}. \quad (3.4)$$

### 3.3 Backward coupling

We are given i.i.d. variables  $(U_0, U_{-1}, U_{-2}, \dots)$ . Any  $U_{-i}$  is sufficient to make a simulation of the Markovian transition above. We introduce a function  $F$  parametrizing this transition (we can write the transition in the following manner:  $(\bar{Z}_1, \dots, \bar{Z}_P) = F_U(z_1, \dots, z_P)$ ). By Theorem 3.1 of [FT98], if  $T$  is a stopping time, relatively to the filtration  $(\sigma(U_0, \dots, U_{-i}))_{i \geq 0}$ , such that  $\forall (z_1^{(1)}, \dots, z_P^{(1)}), (z_1^{(2)}, \dots, z_P^{(2)}) \in E^P, F_{U_{-T}} \circ \dots \circ F_{U_0}(z_1^{(1)}, \dots, z_P^{(1)}) = F_{U_{-T}} \circ \dots \circ F_{U_0}(z_1^{(2)}, \dots, z_P^{(2)})$ , then  $F_{U_{-T}} \circ \dots \circ F_{U_0}(z_1^{(1)}, \dots, z_P^{(1)})$  is exactly of law  $\pi$ .

We now look for a lower bound of (3.4) for a trajectory  $(z_1, \dots, z_P) \in E^P$  and  $i \in \mathbb{N}, U_{-i}$  fixed. We add the following hypothesis.

*Hypothesis 1.* There exists a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that: for all  $x_1 \in E, i \in \mathbb{N}, U_{-i}$  fixed,  $(x_1, \dots, x_P)$  trajectory drawn with transitions  $M$  using the variables  $U_{-i}$  (which we will denote by  $x_{j+1} = M_{U_{-i}}(x_j), \forall j \in \{2, \dots, P\}$ ), for all  $S_\epsilon > 0, \forall j \in \{2, \dots, P\}, \text{diam}(M_{U_{-i}}^{\circ(j-1)}(B_\epsilon(x_1))) \leq f^{\circ(j-1)}(\epsilon)$ .

*Example 3.1.* If the transition  $M$  is (for some constants  $a, b$ ) :

$$M(x, dy) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{(y - ax)^2}{2b^2}\right),$$

then we can take  $f(x) = ax$ .

We now want to bound the number of descendants generated by the trajectory  $(z_1, \dots, z_P)$  during the conditional drawing using the variables  $U_{-i}$ . Let us precise how we do this conditional drawing  $(z_1, \dots, z_P)$ . We fix  $\forall n, \beta_n = \|G_n\|_\infty$  and  $k_n$  satisfying (3.2). For  $g \in [0; \|G\|_\infty]$ , we set  $u \mapsto F_{n,g}^{-1}(u)$  to be pseudo-inverse of the cumulative distribution function of the law  $f_n(g, \cdot)$  and we set  $u \mapsto \widehat{F}_{n,g}^{-1}(u)$  to be the pseudo-inverse of the cumulative distribution function of the law  $\widehat{f}_n(g, \cdot)$ . We are given a family  $(V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n]}, n \geq 1}$  (random variables indexed by infinite sequences of  $\mathbb{N}^*$ ) of independent variables of law  $\mathcal{U}([0; 1])$ . We are given  $(\sigma_{n,N})_{n,N \geq 1}$  independent variables such that  $\forall n, N, \sigma_{n,N}$  is uniform in  $\mathcal{S}_N$ . Suppose there exists  $M' : E \times [0; 1] \rightarrow E$  such that if  $U \sim \mathcal{U}([0; 1]), x \in E$  then  $M'(x, U) \sim M(x, dy)$ . Suppose there exists  $M'_1 : [0; 1] \rightarrow \mathbb{R}$  such that if  $U \sim \mathcal{U}([0; 1]),$  then  $M'_1(U) \sim M_1(dx)$ . The simulation goes as follows.

- We set  $X_1^i = M'_1(V_{(i)})$  ( $(i)$  is a sequence of length 1 taking value  $i$ ) for all  $i \in [N_1] \setminus \{1\}$ , and  $X_1^1 = z_1$ . We define  $\Psi_1 : [N_1] \rightarrow (\mathbb{N}^*)^{[1]}$  by  $\Psi_1(i) = (i)$ .

- Suppose we have made the simulation up to time  $n < P$  and we have a function  $\Psi_n : [N_n] \rightarrow (\mathbb{N}^*)^{[N_n]}$  (describing the genealogy of the particles,  $\Psi_n(i)$  is the complete ancestral line of particle  $i$ ).

- For  $i \in [N_n] \setminus \{1\}$ , we set  $A_{n+1}^i = F_{n, G_n(X_n^i)}^{-1}(W_{\Psi_n(i)})$ ;
- and if  $i = 1$ , then  $X_n^1 = z_n$ ,

and we set  $A_{n+1}^i = \widehat{F}_{n+1, G_n(z_n)}^{-1}(W_{\Psi_n(i)})$ . We set  $N_{n+1} = \sum_{i=1}^{N_n} A_{n+1}^i$ .

- For  $j \in [N_{n+1}] \setminus \{1\}$ , if  $A_{n+1}^1 + \dots + A_{n+1}^{i-1} < j \leq A_{n+1}^1 + \dots + A_{n+1}^i$ , we set  $\Psi_{n+1}(j) = (\Psi_n(i), j - (A_{n+1}^1 + \dots + A_{n+1}^{i-1}))$ ,  $X_{n+1}^j = M'(X_n^i, V_{\Psi_n(j)})$ ,
- and if  $j = 1$ , we set  $X_{n+1}^j = z_{n+1}$ ,  $\Psi_{n+1}(j) = (1, 1, \dots, 1)$ .

- We then set  $\bar{X}_1^i = X_1^{\sigma_{1, N_1}^{i, N_1}}(1)$  ( $1 \leq i \leq N_1$ ),  $B_1 = \sigma_{1, N_1}^{-1}(1)$  (beware,  $B_i$  and  $B_j^i$  have different meanings). We then proceed by recurrence. If we have  $(\bar{X}_j^i)_{1 \leq j \leq n, 1 \leq i \leq N_n}$ ,  $(\bar{A}_j^i)_{2 \leq j \leq n, 1 \leq i \leq N_{j-1}}$ ,  $(\sigma_j)_{2 \leq j \leq n}$ ,  $B_1, \dots, B_n$  with  $\bar{X}_j^i = X_j^{\sigma_{j, N_j}^{i, N_j}}(\forall j \in [n], i \in [N_j])$  then:

We set  $\bar{A}_{n+1}^i = A_{n+1}^{\sigma_{n, N_n}^{i, N_n}}$ ,  $B_{n+1}^i = \{A_{n+1}^1 + \dots + A_{n+1}^{i-1} + 1, \dots, A_{n+1}^1 + \dots + A_{n+1}^i\}$ ,  $\sigma_{n+1} = \sigma_{n+1, N_{n+1}}^{-1}$ ,  $\bar{B}_{n+1}^i = \sigma_{n+1}(B_{n+1}^{\sigma_{n, N_n}^{i, N_n}})$ ,  $\bar{X}_{n+1}^i = X_{n+1}^{\sigma_{n+1, N_{n+1}}^{i, N_{n+1}}}$ , ( $\forall i \dots$ ). We have

- if  $i \in \bar{B}_n^q = \sigma_{n+1, N_{n+1}}^{-1}(B_{n+1}^{\sigma_{n, N_n}^{q, N_n}})$  and  $i \neq B_{n+1} := \sigma_{n+1, N_{n+1}}^{-1}(1)$ ,  $\sigma_{n+1, N_{n+1}}(i) \in B_{n+1}^{\sigma_{n, N_n}^{q, N_n}}$ ,  $X_{n+1}^{\sigma_{n+1, N_{n+1}}(i)} = M'(X_n^{\sigma_{n, N_n}^{q, N_n}}, V_{\Psi_n(\sigma_{n, N_n}^{q, N_n})})$ , then  $\bar{X}_{n+1}^i = M'(\bar{X}_n^q, V_{\Psi_n(\sigma_{n, N_n}^{q, N_n})})$
- and in the case  $i = B_{n+1}$ ,  $\bar{X}_{n+1}^{B_{n+1}} = X_{n+1}^1 = z_{n+1}$ .

And we have

- if  $B_{n+1} \notin \bar{B}_{n+1}^i$ , then  $\# \bar{B}_{n+1}^i = \# B_{n+1}^{\sigma_{n, N_n}^{i, N_n}} = A_{n+1}^{\sigma_{n, N_n}^{i, N_n}} = F_{n, G_n(\bar{X}_n^i)}^{-1}(W_{\Psi_n(\sigma_{n, N_n}^{i, N_n})})$ ,
- if  $B_{n+1} \in \bar{B}_{n+1}^i$ , then  $\# \bar{B}_{n+1}^i = \widehat{F}_{n, G_n(\bar{X}_n^i)}^{-1}(W_{\Psi_n(\sigma_{n, N_n}^{i, N_n})})$ .

This procedure draw  $(\bar{X}_n^i, \bar{A}_n^i, B_n, \sigma_n)$  with the density (3.3) (in practice, one can get rid of the simulation of the permutations since they have no influence on the trajectories we are interested in). We will note  $(X_n^i, A_n^i, B_n, \sigma_n, n \in \dots) = \Phi((z_i)_{i \in [P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n], n \geq 1}}, (G_n)_{1 \leq n \leq P})$ .

*Lemma 3.2.* If in the procedure above, we replace  $A_{n+1}^i = \widehat{F}_{n+1, G_n(z_n)}^{-1}(W_{\Psi_n(i)})$  (in the case  $\Psi_n(i) = (N_1, 1, \dots, 1)$ ) by  $\tilde{A}_{n+1}^i = \widehat{F}_{n+1, H_n(z_n)}^{-1}(W_{\Psi_n(i)})$  for some function  $H_n \geq G_n$ , then we get a different system, which we will note with  $\tilde{\phantom{x}}$ ,

$$(\tilde{X}_n^i, \tilde{A}_n^i, \tilde{B}_n, \tilde{\sigma}_n, n \in \dots) = \Phi((z_i)_{i \in [P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n], n \geq 1}}, (H_n)_{1 \leq n \leq P}),$$

such that  $\forall n, \{X_n^i, 1 \leq n \leq N_n\} \subset \{\tilde{X}_n^i, 1 \leq n \leq \tilde{N}_n\}$ . moreover, the descendants of  $z_1, \dots, z_P$  at time  $P$  are independent variables.

Let  $\delta > 0$ . For all  $n \in [P]$ , let us take  $H_1 = G_1, \dots, H_{n-1} = G_{n-1}$ , and for  $k \geq n$ ,

$$H_k(x) = \begin{cases} \sup_{|y - z_k| < f^{\circ(k-n)}(\delta)} G_n(y) & \text{if } |x - z_k| \leq f^{\circ(k-n)}(\delta) \\ G_n(y) & \text{otherwise,} \end{cases}$$

and let us note with  $\tilde{\phantom{x}}$  the corresponding system,

$$\text{meaning } (\tilde{X}_n^i, \tilde{A}_n^i, \tilde{B}_n, \tilde{\sigma}_n, n \in \dots) = \Phi((z_i)_{i \in [P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n], n \geq 1}}, (H_n)_{1 \leq n \leq P}).$$

Let  $z'_1, \dots, z'_P$  be such that  $z'_i \in B_\delta(z_i), \forall i$ . We have

$$(X_n^i, A_n^i, B_n, \sigma_n, n \in \dots) = \Phi((z'_i)_{i \in [P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n]}, n \geq 1}, (G_n)_{1 \leq n \leq P}).$$

Using the Lemma above and Hypothesis 1, we have  $N_P \leq \tilde{N}_P$ . Let  $\Phi'$  be such that

$$N_P = \Phi'((z_i)_{i \in [P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[n]}, n \geq 1}, (H_n)_{1 \leq n \leq P}).$$

### 3.4 Examples

#### 3.4.1 Gaussian example

We draw sequences  $(X_n)_{n \in [P]}, (Y_n)_{n \in [P]}$  such that:  $X_1 \sim \mathcal{N}(0, 1), X_{n+1} = aX_n + bV_{n+1} (a \in ]0; 1[), Y_n = X_n + cW_n$  with i.i.d. variables  $V_n, W_n$  of law  $\mathcal{N}(0, 1)$ . We set

$$G_n(x) = \frac{1}{\sqrt{2\pi c^2}} \exp\left(-\frac{1}{2c^2}(x - Y_n)^2\right),$$

$M_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, M(x, dy) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{(y-ax)^2}{2b^2}\right) dy$ . We want to bound, at time  $P$ , the particles descending from a fixed trajectory. The descendants of different  $z_n$  are independent so we look, for all  $n$ , at which is the  $z_n$  spawning the most descendants at time  $P$ . Using the result above, we slice  $E$  in balls of size  $\delta$ . If  $z'_n$  is in a ball of size  $\delta$  containing  $z_n$ , the number of descendants of  $z'_n$  at time  $P$  computed with potentials  $G$ , is bounded by the number of descendants of  $z_n$  at time  $P$  computed with potentials  $H$ . The potentials  $G_n$  going to 0 at  $\pm\infty$ , we do not have to explore the whole of  $\mathbb{R}$ , as soon as  $z_n$  is far enough from  $Y_n$  so that it has 0 children under potential  $H_n$ , we can stop the exploration.

*Remark 3.1.* With  $\delta = 0$  (or  $\delta$  very small), if we look at the number of descendants at time  $P$  of an individual at time  $n$  and we maximise in the position of the individual, we will find some finite quantity (not exploding when  $P - n \rightarrow +\infty$ ). For the maximisation step, we have to take  $\delta > 0$  and then this maximum explodes (slowly). So, there is a balance to find between  $\delta$  small (maximisation step takes a lot of time) and  $\delta$  big (explosion in the number of particles). A rule of thumb, coming from the experience, is that the population does not explode as long as the number of children per individual is of order 2, 3.

#### 3.4.2 Directed polymers

Let  $(X_n)_{n \geq 1}$  be a symmetric simple random walk in  $\mathbb{Z}$  with  $X_1 = 0$ . We are given i.i.d. variables  $(\xi_{n,i})_{n \geq 1, i \in \mathbb{Z}}$  with Bernoulli law of parameter  $p > 0$ . We set (random) potentials:  $V_n(i) = \exp(-\beta \xi_{n,i})$  ( $\beta > 0$ ) and we are interested in the following law (quenched, meaning the  $\xi_{n,i}$  are fixed):

$$\pi_{1:n}(f) = \frac{\mathbb{E}_\xi(f(X_{1:n}) \prod_{k=1}^n V_k(X_k))}{\mathbb{E}_\xi(\prod_{k=1}^n V_k(X_k))}.$$

This kind of model is described in [BTV08]. If we take the expectation over all the variables:  $\mathbb{E}(\max \text{ de la traj. sous } \pi_{1:n})$  behaves as  $n^\zeta$  with  $\zeta \neq 1/2$ .

Using our algorithm, we can simulate trajectories under the law  $\pi_{1:P}$  (for fixed  $\xi, P \in \mathbb{N}^*$ ). The research of the ancestors having the biggest number of descendants at time  $P$  makes that the computational cost is  $P^2$ . Here is the drawing of  $\mathbb{E}(\max \dots)$  as a function of  $n$  in a log-log scale (the blue line has gradient  $2/3$ , the green line has gradient  $1/2$ ):

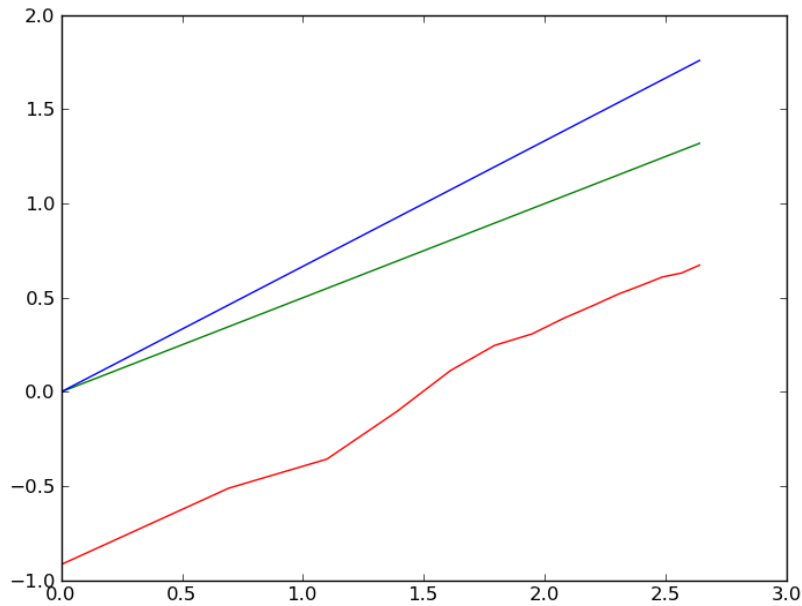


Figure 3.1: gradient estimation (least square)=0,63

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