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Identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract

Improving an old idea of Hermite by using the Laplace-Borel transform, we present a new class of identities linking Cauchy numbers, harmonic numbers and zeta values.

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Keywords : Cauchy numbers, Bell polynomials, Harmonic numbers, Laplace-Borel transform, Mellin transform, Zeta values.

1 Introduction

It is well known since the second-half of the 19th century that the Riemann zeta function may be represented by the (normalized) Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} dt \quad \text{for } \Re(s) > 1,$$

and from late works of Hermite (cf. [8]) that one has also

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \left(\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^n \right) dt \quad \text{for } \Re(s) \geq 1,$$

where $\lambda_1 = \frac{1}{2}$ and $\lambda_{n+1} = \int_0^1 x(1-x)\cdots(n-x) dx$ are the (non-alternating) Cauchy numbers.

Improving Hermite's idea, one may, more generally, consider Mellin transforms of type

$$F(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f(1-e^{-t}) dt$$

with $f(z) = \sum_{n=1}^{\infty} \omega_n \frac{z^n}{n^k}$ for suitable sequences $(\omega_n)_{n \geq 1}$ of rational numbers. The simplest interesting case $\omega_n = 1$ has been studied in [6]. In this article, we investigate the case $\omega_n = \frac{\lambda_n}{n!}$ *i.e.* we study the function

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t}) dt \text{ with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k} \quad (k = 0, 1, 2, \dots),$$

which is *a priori* defined in the half-plane $\Re(s) \geq 1$ but analytically continues in the whole complex s -plane (Theorem 6). For $k = 0$, one must keep in mind that $F_0(s)$ is nothing else than $\zeta(s) - \frac{1}{s-1}$.

An evaluation of the values of F_k at positive integers $q \geq 2$ by two different ways (Theorem 4 and Theorem 7) leads to a class of new identities linking Cauchy numbers, harmonic numbers and zeta values (Theorem 8). For $k = 0$, one recovers *Hermite's formula* (cf. [5]),

$$F_0(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \zeta(q) - \frac{1}{q-1},$$

where the polynomials P_m are the *modified Bell polynomials* defined by the generating function

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m}\right) = \sum_{m=0}^{\infty} P_m(x_1, \dots, x_m) z^m,$$

and $H_n^{(m)}$ are the harmonic numbers. For $k = 1$, one obtains the following relation

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}},$$

For example, in the simplest case $q = 2$, one has $P_1(H_n) = H_n$, and $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$;

hence, the previous relations may be written

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} &= \zeta(2) - 1, \\ \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2} &= \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1.\end{aligned}$$

The function F_k has also an interesting interpretation in terms of Ramanujan summation (cf. [3]) as underscored by Theorem 11. In particular, one shows the identity

$$F_k(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n \geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{n}$$

where, in the right member, $\sum_{n \geq 1}^{\mathcal{R}}$ denotes the sum (in the sense of Ramanujan) of the divergent series. This raises a kind of “duality” between $F_k(1)$ and $F_0(k+1)$.

2 Preliminaries

2.1 The non-alternating Cauchy numbers

Definition 1. The *Cauchy numbers* ([5], [9]) are the rational numbers \mathcal{C}_m defined for all natural numbers m by the exponential generating function :

$$\sum_{m \geq 0} \mathcal{C}_m \frac{z^m}{m!} = \frac{z}{\log(1+z)}.$$

Let $\lambda_{n+1} := (-1)^n \mathcal{C}_{n+1}$, then $\lambda_{n+1} > 0$, and changing z in $-z$, we get the following relation

$$\frac{1}{\log(1-z)} + \frac{1}{z} = \sum_{n \geq 0} \frac{\lambda_{n+1}}{(n+1)!} z^n. \quad (1)$$

For $z = 1 - e^{-t}$ and $t > 0$, this relation may be rewritten

$$\frac{1}{1 - e^{-t}} - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1}. \quad (2)$$

For each integer $n \geq 1$, we will call λ_n the *n*th *non-alternating* Cauchy number.

Example 1. The first non-alternating Cauchy numbers are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{1}{4}, \lambda_4 = \frac{19}{30}, \lambda_5 = \frac{9}{4}.$$

2.2 The modified Bell polynomials and the harmonic numbers

Definition 2. The *modified Bell polynomials* (cf. [5], [7]) are the polynomials P_m defined for all natural numbers m by the generating function

$$\exp\left(\sum_{m \geq 1} x_m \frac{z^m}{m}\right) = \sum_{m \geq 0} P_m(x_1, \dots, x_m) z^m. \quad (3)$$

Proposition 1. For all natural numbers m , and each integer $n \geq 1$,

$$\int_0^{+\infty} e^{-t}(1 - e^{-t})^{n-1} \frac{t^m}{m!} dt = \frac{P_m(H_n, \dots, H_n^{(m)})}{n} \quad (4)$$

with

$$H_n^{(m)} := \sum_{j=1}^n \frac{1}{j^m} \quad \text{and} \quad H_n := H_n^{(1)}.$$

Proof. One starts from the classical Euler's relation :

$$B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and substitute $u = e^{-t}$, $a = 1 - z$ and $b = n + 1$, then one obtains

$$\int_0^{+\infty} e^{-t}(1 - e^{-t})^n e^{tz} dt = \frac{n!}{(1-z)(2-z)\dots(n+1-z)}.$$

Moreover, one has

$$\begin{aligned} \frac{n!}{(1-z)(2-z)\dots(n+1-z)} &= \frac{n!}{(n+1)!} \times \prod_{k=0}^n \left(1 - \frac{z}{k+1}\right)^{-1} \\ &= \frac{1}{(n+1)} \times \exp\left(-\sum_{k=0}^n \log\left(1 - \frac{z}{k+1}\right)\right) \\ &= \frac{1}{(n+1)} \times \exp\left(\sum_{k=0}^n \sum_{m=1}^{\infty} \frac{z^m}{m(k+1)^m}\right) \\ &= \frac{1}{(n+1)} \exp\left(\sum_{m=1}^{\infty} H_{n+1}^{(m)} \frac{z^m}{m}\right). \end{aligned}$$

Thus, by identification, (4) follows directly from (3). \square

Example 2. For small values of m , one has

$$\begin{aligned} P_0 &= 1; \quad P_1(H_n) = H_n; \quad P_2(H_n, H_n^{(2)}) = \frac{(H_n)^2}{2} + \frac{H_n^{(2)}}{2}; \\ P_3(H_n, H_n^{(2)}, H_n^{(3)}) &= \frac{(H_n)^3}{6} + \frac{H_n H_n^{(2)}}{2} + \frac{H_n^{(3)}}{3}. \end{aligned}$$

2.3 The Laplace-Borel transformation

We consider the vector space E of complex-valued functions $f \in \mathcal{C}^1(]0, +\infty[)$ such that

for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|f(t)| \leq C_\varepsilon e^{\varepsilon t}$ for all $t \in]0, +\infty[$.

In particular, a function $f \in E$ satisfies the two following properties :

- a) for all x with $\Re(x) > 0$, $t \mapsto e^{-xt} f(t)$ is integrable on $]0, +\infty[$,
- b) for all β with $0 < \beta < 1$, $t \mapsto |f(t)| \frac{1}{t^\beta}$ is integrable on $]0, 1[$.

We recall now some basic properties (cf. [10]) of the Laplace transformation in this frame which is appropriate for our purpose.

Definition 3. Let f be a function in E . The *Laplace transform* $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(x) = \int_0^{+\infty} e^{-xt} f(t) dt \quad \text{for } \Re(x) > 0.$$

Proposition 2 (cf. [10]). Let $\mathcal{E} := \mathcal{L}(E)$ be the image of E under \mathcal{L} . If a is a function in \mathcal{E} , then

- a) a is an analytic function of x in the half-plane $\Re(x) > 0$.
- b) $a(x) \rightarrow 0$ when $\Re(x) \rightarrow +\infty$.
- c) $\mathcal{L} : E \rightarrow \mathcal{E}$ is an isomorphism.

Definition 4. Let $a \in \mathcal{E}$. The *Borel transform* of a is the unique function $\hat{a} \in E$ such that $a = \mathcal{L}(\hat{a})$. One has the two reciprocal formulas

$$\hat{a}(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{zt} a(z) dz \quad \text{for all } c > 0 \text{ and } t > 0,$$

and

$$a(x) = \int_0^{+\infty} e^{-xt} \hat{a}(t) dt \quad \text{for } \Re(x) > 0.$$

Proposition 3 (cf. [10]). If $f \in E$ and $g \in E$, then $f * g \in E$ and

$$\mathcal{L}(f * g) = \mathcal{L}(f) \mathcal{L}(g). \tag{5}$$

Hence, if $a \in \mathcal{E}$ and $b \in \mathcal{E}$ then $ab \in \mathcal{E}$ since $ab = \mathcal{L}(\hat{a} * \hat{b})$.

Theorem 1. Let a be a function in \mathcal{E} . Then the series

$$\sum_{n \geq 1} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} (1 - e^{-t})^{n-1} \hat{a}(t) dt$$

converges and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} (1 - e^{-t})^{n-1} \hat{a}(t) dt = \int_0^{+\infty} \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \hat{a}(t) dt. \tag{6}$$

Proof. By (2)

$$\int_0^{+\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t} \widehat{a}(t) dt = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^{n-1} e^{-t} \widehat{a}(t) dt.$$

In the right member, the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since

$$\begin{aligned} \int_0^{+\infty} \sum_{n=1}^{\infty} \left| \frac{\lambda_n}{n!} (1-e^{-t})^{n-1} e^{-t} \widehat{a}(t) \right| dt &= \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^{n-1} e^{-t} |\widehat{a}(t)| dt \\ &= \int_0^{+\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t} |\widehat{a}(t)| dt \end{aligned}$$

and the convergence of this last integral follows from the assumption that $a \in \mathcal{E}$. \square

Example 3. Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \geq 1$. Then $a \in \mathcal{E}$ and $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$. Hence

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} (1-e^{-t})^{n-1} \frac{t^{s-1}}{\Gamma(s)} dt = \int_0^{+\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t} \frac{t^{s-1}}{\Gamma(s)} dt = \begin{cases} \gamma & \text{if } s = 1 \\ \zeta(s) - \frac{1}{s-1} & \text{if } s \neq 1 \end{cases}$$

where γ refers to the Euler constant. In particular, since

$$\int_0^{+\infty} e^{-t} (1-e^{-t})^{n-1} dt = \frac{1}{n} \quad \text{for each integer } n \geq 1,$$

then

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n}.$$

3 The operator D

Proposition 4. If $a \in \mathcal{E}$, then the integral

$$\int_0^{+\infty} e^{-t} (1-e^{-t})^{x-1} \widehat{a}(t) dt$$

converges for all x with $\Re(x) > 0$.

Proof. If $a \in \mathcal{E}$ and $\Re(x) > 0$, we may write for $t \in]0, +\infty[$,

$$\left| e^{-t} (1-e^{-t})^{x-1} \widehat{a}(t) \right| \leq e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} |\widehat{a}(t)|.$$

The convergence when $t \rightarrow +\infty$ results from the inequality

$$e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} |\widehat{a}(t)| \leq \frac{e^{-t}}{1-e^{-t}} |\widehat{a}(t)| \leq 2e^{-t} |\widehat{a}(t)|.$$

The convergence when $t \rightarrow 0$ results from the inequality

$$e^{(1-\Re(x))(-\log(1-e^{-t}))} \leq \begin{cases} 1 & \text{si } \Re(x) \geq 1 \\ \frac{1}{(1-e^{-t})^{1-\Re(x)}} & \text{si } 0 < \Re(x) < 1 \end{cases}$$

since the function $t \mapsto e^{-t} |\widehat{a}(t)| \frac{1}{(1-e^{-t})^\beta}$ is integrable at 0 for $0 < \beta < 1$ by definition of E . \square

Definition 5. Let a be a function in \mathcal{E} . We call $D(a)$ the function defined for all x with $\Re(x) > 0$ by

$$D(a)(x) = \int_0^{+\infty} e^{-t} (1 - e^{-t})^{x-1} \widehat{a}(t) dt. \quad (7)$$

Remark 1. a) By Theorem 1, the series $\sum_{n \geq 1} \frac{\lambda_n}{n!} D(a)(n)$ converges and its sum is given by formula (6).

b) The values of $D(a)$ at positive integers may be computed directly without the recourse to \widehat{a} . The development of $(1 - e^{-t})^n$ by the binomial theorem gives

$$D(a)(n+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} a(k+1) \quad \text{for all integer } n \geq 0. \quad (8)$$

Definition 6. We call Λ the C^1 -diffeomorphism of \mathbb{R}_+ defined by $\Lambda(u) := -\log(1 - e^{-u})$. In particular, it is important to note that Λ is involutive :

$$\Lambda^{-1} = \Lambda.$$

Theorem 2. Let a be a function in \mathcal{E} . Then the function $D(a) \in \mathcal{E}$ and, moreover, verifies the relation

$$\widehat{D(a)} = \widehat{a}(\Lambda) \quad (9)$$

where $\widehat{a}(\Lambda)$ denotes $\widehat{a} \circ \Lambda$.

Proof. The change of variables $t = \Lambda(u)$ in (7) gives

$$D(a)(x) = \int_0^{+\infty} e^{-xu} \widehat{a}(\Lambda(u)) du \quad \text{for } \Re(x) > 0.$$

Thus, $D(a) = \mathcal{L}(\widehat{a}(\Lambda))$. It remains to prove that $D(a) \in \mathcal{E}$. One has only to check that the function $\widehat{a}(\Lambda)$ is in E . This function being in $C^1(]0, +\infty[)$, it suffices to show that for all $\varepsilon > 0$, the function $u \mapsto e^{-\varepsilon u} |\widehat{a}(-\log(1 - e^{-u}))|$ is bounded on $]0, +\infty[$. This results from the existence of $C_\varepsilon > 0$ such that

$$|\widehat{a}(-\log(1 - e^{-u}))| \leq C_\varepsilon (1 - e^{-u})^\varepsilon \quad \text{for all } u \in]0, +\infty[.$$

\square

Example 4. Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \geq 1$. Then $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$. Thus, by (9),

$$D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right). \quad (10)$$

If m is a natural number and $s = m + 1$, then by (4) and (7)

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \mathcal{L}\left(\frac{\Lambda^m}{m!}\right)(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}. \quad (11)$$

By (8), one has also

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} \frac{1}{k^{m+1}}.$$

Thus, from (11) and *Dilcher's formula* (cf. [2] Proposition 11), one deduces the nice identity

$$P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k^m} = \sum_{n \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1 \dots n_m}. \quad (12)$$

Remark 2. Theorem 2 may be summarized in the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \mathcal{E} \\ \downarrow \mathcal{L}^{-1} & & \uparrow \mathcal{L} \\ E & \xrightarrow{\Lambda^*} & E \end{array}$$

where $\Lambda^*(\widehat{a}) := \widehat{a}(\Lambda)$. The algebraic properties of D are sum up in the following theorem.

Theorem 3. *The operator D is an automorphism of \mathcal{E} which verifies $D = D^{-1}$ and lets invariant the function $x \mapsto \frac{1}{x}$.*

Proof. We can write $D = \mathcal{L}\Lambda^*\mathcal{L}^{-1}$ and Λ^* is an automorphism of E which verifies $\Lambda^* = (\Lambda^*)^{-1}$ since $\Lambda = \Lambda^{-1}$. Furthermore

$$D\left(\frac{1}{x}\right) = \mathcal{L}(1) = \frac{1}{x}.$$

□

4 The function F_k

4.1 Series representation

Theorem 4. For all s in \mathbb{C} with $\Re(s) \geq 1$ and each natural number k , let

$$F_k(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^k} D\left(\frac{1}{x^s}\right)(n). \quad (13)$$

Then, for all natural numbers m ,

$$F_k(m+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^{k+1}} P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}). \quad (14)$$

Proof. By (11), one has $D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}$. \square

Remark 3. Since $F_0(s) = \zeta(s) - \frac{1}{s-1}$, then, in the case $k = 0$, (14) is nothing else than *Hermite's formula* for ζ (cf. [5]).

Corollary 1. Let $\vartheta(s)$ be the Dirichlet series defined for $\Re(s) > 0$ by

$$\vartheta(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^s}.$$

Then for each natural number $k \geq 1$,

$$\vartheta(k) = F_{k-1}(1). \quad (15)$$

Remark 4. By (1) and a tauberian theorem, one has $\vartheta(0) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} = 1$.

Example 5.

$$\begin{aligned} F_0(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \gamma = \vartheta(1), \\ F_0(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} = \zeta(2) - 1, \\ F_0(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n} = \zeta(3) - \frac{1}{2}, \\ F_1(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \vartheta(2), \\ F_1(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2}, \\ F_1(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2}. \end{aligned}$$

4.2 Integral representation

Theorem 5. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k ,

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t}) dt \quad \text{with} \quad f_k(z) := \sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n! n^k}. \quad (16)$$

Proof. Since $D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)$, we deduce from (13) that

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_k(e^{-u}) (\Lambda(u))^{s-1} du$$

and the representation (16) results from the change of variables $t = \Lambda(u)$. \square

The fact that F_k may be represented by a Mellin transform enables to analytically continue this function outside its half-plane of definition by a standard analytic method (cf. [12] section 6.7).

Theorem 6. The function F_k analytically continues in the whole complex plane as an entire function.

Proof. The function $z \mapsto \frac{1}{\log(1-z)} + \frac{1}{z}$ being analytic in the disc $D(0, 1)$ with a singularity at 1, we deduce from (1) that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n!}$ is equal to 1. Thus 1 is also the radius of convergence of the serie $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n! n^k}$ which defines an analytic function f_k in the disc $D(0, 1)$. Hence, the function

$$g_k : t \mapsto f_k(1 - e^{-t})$$

is analytic for all $t \in \mathbb{C}$ such that $1 - e^{-t} \in D(0, 1)$. Since $1 - e^0 = 0$, it follows that g_k is analytic in a neighbourhood of 0. Since $g_k(0) = 0$, the function $t \mapsto g_k(t) \frac{e^{-t}}{1 - e^{-t}}$ is itself analytic in a neighbourhood of 0. It follows that its Mellin transform analytically continues in the complex plane with simple poles at negative integers which are all cancelled by the poles of Γ . \square

Theorem 7. For all s with $\Re(s) > 1$ and each integer $k \geq 1$,

$$F_k(s) = \vartheta(k)\zeta(s) + \sum_{j=1}^k (-1)^j \vartheta(k-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt \quad (17)$$

with

$$Z_j(s) := \sum_{n > n_1 > n_2 > \dots > n_j > 0} \frac{1}{n^s n_1 n_2 \dots n_j} \quad (18)$$

and T is the operator defined by

$$Tf(t) := \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} f(u) du. \quad (19)$$

Proof. The theorem results from the integral representation (16) and the two following lemmas.

Lemma 1. Let T be the operator defined by (19). Then for all $t > 0$,

$$f_k(1 - e^{-t}) = \sum_{j=0}^k (-1)^j \vartheta(k-j) \frac{\Lambda^j(t)}{j!} + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right).$$

Proof. Let $g_k(t) := f_k(1 - e^{-t})$. The function g_k verifies the recursive relation

$$g'_k(t) = e^{-t} f'_k(1 - e^{-t}) = \frac{e^{-t}}{1 - e^{-t}} f_{k-1}(1 - e^{-t}) = \frac{e^{-t}}{1 - e^{-t}} g_{k-1}(t)$$

Thus

$$g_k(t) = \int_0^t \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du = g_k(+\infty) - \int_t^{+\infty} \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du$$

with

$$g_k(+\infty) = f_k(1) = \vartheta(k).$$

Thus, one has

$$g_k(t) = \vartheta(k) - \int_t^{+\infty} \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du = \vartheta(k) - T(g_{k-1}).$$

A repeated iteration k times of this relation gives

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j) (-1)^j T^j(1) + (-1)^k T^k(g_0).$$

Now, by (2),

$$g_0(t) = \sum_{n=1}^{\infty} \frac{\lambda_n (1 - e^{-t})^n}{n!} = \frac{e^{-t} - 1}{t} + 1,$$

and thus

$$T^k(g_0) = T^k\left(\frac{e^{-t} - 1}{t}\right) + T^k(1).$$

Hence

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j) (-1)^j T^j(1) + (-1)^k T^k(1) + (-1)^k T^k\left(\frac{e^{-t} - 1}{t}\right).$$

Since $\vartheta(0) = 1$, one deduces that

$$g_k(t) = \sum_{j=0}^k \vartheta(k-j)(-1)^j T^j(1) + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right)$$

and, now, it remains to prove that

$$\frac{\Lambda^j(t)}{j!} = T^j(1)$$

which follows from the recursive relation

$$\frac{\Lambda^j(t)}{j!} = - \int_{+\infty}^t \frac{e^{-u}}{1-e^{-u}} \frac{\Lambda^{j-1}(u)}{(j-1)!} du = T\left(\frac{\Lambda^{j-1}}{(j-1)!}\right).$$

□

Lemma 2. Let $Z_j(s)$ defined by (18). Then, for all $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$Z_j(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt.$$

Proof. From the recursive relation

$$\partial \frac{\Lambda^j(t)}{j!} = \frac{\Lambda^{j-1}(t)}{(j-1)!} \partial \Lambda(t) = - \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^{j-1}(t)}{(j-1)!} = - \sum_{m>0} e^{-mt} \frac{\Lambda^{j-1}(t)}{(j-1)!},$$

and $\Lambda(t) = \sum_{n>0} \frac{e^{-nt}}{n}$, one may check by induction on j that

$$\frac{\Lambda^j(t)}{j!} = \sum_{n_1 > n_2 > \dots > n_j > 0} \frac{e^{-n_1 t}}{n_1} \frac{1}{n_2} \dots \frac{1}{n_j}.$$

Furthermore, one has

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-Nt} \frac{e^{-t}}{1-e^{-t}} dt = \sum_{n>N} \frac{1}{n^s} \quad (\text{for } \Re(s) > 1).$$

Hence

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt = \sum_{n > n_1 > n_2 > \dots > n_j > 0} \frac{1}{n^s} \frac{1}{n_1} \frac{1}{n_2} \dots \frac{1}{n_j} = Z_j(s).$$

□

□

4.3 Identities linking Cauchy numbers, harmonic numbers and zeta values

From Theorem 4 and Theorem 7, we immediately deduce the following theorem.

Theorem 8. *For all integers $q \geq 2$,*

$$F_0(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \zeta(q) - \frac{1}{q-1}, \quad (20)$$

and for $k \geq 1$,

$$\begin{aligned} F_k(q) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^{k+1}} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \\ &\vartheta(k)\zeta(q) + \sum_{j=1}^k (-1)^j \vartheta(k-j) Z_j(q) + (-1)^k \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt. \end{aligned} \quad (21)$$

Corollary 2. *For all integers $q \geq 2$,*

$$\begin{aligned} F_1(q) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \\ &\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}. \end{aligned} \quad (22)$$

Proof. We apply (21) with $k = 1$. This gives

$$F_2(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_{n-1}}{n^q} + \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathbf{E}_1(t) dt$$

with $\mathbf{E}_1(t) := -\text{Ei}(-t) = \int_t^{+\infty} \frac{e^{-u}}{u} du$. Thus,

$$F_2(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_n}{n^q} + \zeta(q+1) + I(q)$$

where

$$I(q) = \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathbf{E}_1(t) dt = \frac{1}{\Gamma(q)} \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-nt} t^{q-1} \mathbf{E}_1(t) dt.$$

Since

$$\mathbf{E}_1(t) = -\gamma - \log t + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n n!},$$

and $-\gamma - \log t = \widehat{\frac{\log x}{x}}$ (cf. [10]), then $E_1 = \widehat{\frac{\log(x+1)}{x}}$. Thus

$$\int_0^{+\infty} e^{-nt} t^{q-1} E_1(t) dt = (-1)^{q-1} \left(\frac{\log(x+1)}{x} \right)^{(q-1)} (n).$$

Hence, by a calculation of the $(q-1)$ th derivative, we get

$$I(q) = \frac{(-1)^{q-1}}{(q-1)!} \sum_{n=1}^{\infty} \left(\frac{\log(x+1)}{x} \right)^{(q-1)} (n) = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}.$$

□

Remark 5. 1) We recall *Euler's formula* (cf. [1], [11])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \text{ and } \sum_{n=1}^{\infty} \frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2} \sum_{k=1}^{q-2} \zeta(k+1)\zeta(q-k) \text{ for } q > 2.$$

2) From $\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$ and the decomposition

$$\frac{1}{(n+1)^k n^{q-k}} = \frac{1}{(n+1)^{k-1} n^{q-k}} - \frac{1}{(n+1)^k n^{q-k-1}} \quad (0 < k < q),$$

the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}$ may be expressed as a linear combination of zeta values and integers.

Example 6.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1 &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2}, \\ \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^3} + \gamma\zeta(3) - \frac{1}{10}\zeta(2)^2 - \frac{1}{2}\zeta(2) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2}, \\ \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^4} + \gamma\zeta(4) - 2\zeta(5) + \zeta(2)\zeta(3) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2} &= \\ \frac{1}{6} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^3}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n H_n^{(2)}}{n!n^2} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(3)}}{n!n^2}. \end{aligned}$$

4.4 Link with the Ramanujan summation

The function F_k has strong connections with Ramanujan summation (cf. [3], [4]).

Definition 7. Let a be a function in $\mathcal{E} = \mathcal{L}(E)$. The *Ramanujan sum* of the series $\sum_{n \geq 1} a(n)$ is defined by

$$\sum_{n \geq 1}^{\mathcal{R}} a(n) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D(a)(n). \quad (23)$$

Proposition 5. If a and b are in \mathcal{E} , then $\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \in E$.

Proof. From the definition of the convolution product, one may write

$$\left(\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \right) (t) = \int_0^t \widehat{a}(\Lambda(u)) \widehat{b}(\Lambda(t-u)) du.$$

Now, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $D_\varepsilon > 0$ such that

$$\begin{aligned} \left| \widehat{a}(-\log(1 - e^{-u})) \right| &\leq C_\varepsilon (1 - e^{-u})^\varepsilon \text{ and} \\ \left| \widehat{b}(-\log(1 - e^{-(t-u)})) \right| &\leq D_\varepsilon (1 - e^{-(t-u)})^\varepsilon \text{ for all } u \in]0, +\infty[. \end{aligned}$$

It follows that

$$\left| (\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t) \right| \leq C_\varepsilon D_\varepsilon \int_0^t (1 - e^{-u})^\varepsilon (1 - e^{-(t-u)})^\varepsilon du.$$

One has also

$$\begin{aligned} \int_0^t (1 - e^{-u})^\varepsilon (1 - e^{-(t-u)})^\varepsilon du &= (1 - e^{-t})^{1+2\varepsilon} \int_0^1 u^\varepsilon (1-u)^\varepsilon \frac{1}{(1 - (1 - e^{-t})u)^{\varepsilon+1}} du \\ &\leq (1 - e^{-t})^{1+2\varepsilon} \int_0^1 \frac{1}{(1 - (1 - e^{-t})u)^{\varepsilon+1}} du \leq (1 - e^{-t})^{1+2\varepsilon} \frac{e^{t\varepsilon} - 1}{(1 - e^{-t})^\varepsilon} \\ &\leq (1 - e^{-t})^{2\varepsilon} \frac{e^{t\varepsilon} - 1}{\varepsilon} \leq \frac{e^{t\varepsilon}}{\varepsilon}. \end{aligned}$$

Hence, $\left| (\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t) \right| \leq C_\varepsilon D_\varepsilon \frac{e^{t\varepsilon}}{\varepsilon}$, which proves that this function belongs to E as required. \square

Definition 8. Let a and b two functions in \mathcal{E} . The Λ -convolution product $\widehat{a} \circledast \widehat{b}$ of \widehat{a} and \widehat{b} is defined by

$$\widehat{a} \circledast \widehat{b} = \Lambda^*(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b}))$$

(or equivalently since $\Lambda^* = (\Lambda^*)^{-1}$)

$$(\widehat{a} \circledast \widehat{b})(\Lambda) = \widehat{a}(\Lambda) * \widehat{b}(\Lambda),$$

and the *harmonic product* $a \bowtie b$ of a and b by

$$a \bowtie b = \mathcal{L}(\widehat{a} \circledast \widehat{b}).$$

Remark 6. The Λ -convolution product and the harmonic product inherit of the algebraic properties of the ordinary convolution product *i.e.* bilinearity, commutativity and associativity. This construction may be summarized in the following diagram

$$\begin{array}{ccccc} (a, b) & \longrightarrow & (\widehat{a}, \widehat{b}) & \longrightarrow & (\widehat{a}(\Lambda), \widehat{b}(\Lambda)) \\ \downarrow & & \downarrow & & \downarrow \\ a \rtimes b & \longleftarrow & \widehat{a} \otimes \widehat{b} & \longleftarrow & \widehat{a}(\Lambda) * \widehat{b}(\Lambda) \end{array}$$

Theorem 9. *Let a and b in \mathcal{E} . Then,*

$$D(a \rtimes b) = D(a) D(b) \tag{24}$$

and

$$D(ab) = D(a) \rtimes D(b). \tag{25}$$

Proof. One recalls (cf. Theorem 3) that

$$D = \mathcal{L}\Lambda^*\mathcal{L}^{-1}.$$

Hence

$$D(a \rtimes b) = \mathcal{L}\Lambda^*\mathcal{L}^{-1}(a \rtimes b) = \mathcal{L}\Lambda^*(\widehat{a} \otimes \widehat{b}) = \mathcal{L}(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b}))$$

and it follows from (5) and (9) that

$$\mathcal{L}(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b})) = \mathcal{L}(\Lambda^*(\widehat{a}))\mathcal{L}(\Lambda^*(\widehat{b})) = D(a) D(b)$$

which proves (24). Moreover, (24) enables to write

$$D(D(a) \rtimes D(b)) = D^2(a) D^2(b) = ab \quad (\text{since } D = D^{-1}),$$

and so

$$D(ab) = D^2(D(a) \rtimes D(b)) = D(a) \rtimes D(b)$$

which proves (25). □

Corollary 3. *Let a and b in \mathcal{E} . Then*

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} (a \rtimes b)(n) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D(a)(n) D(b)(n), \\ \sum_{n \geq 1}^{\mathcal{R}} (ab)(n) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (D(a) \rtimes D(b))(n). \end{aligned} \tag{26}$$

Remark 7. The values of $(a \rtimes b)(n)$ may be computed without the recourse to \widehat{a} and \widehat{b} . By elementary transformations, it can be shown that

$$(a \rtimes b)(n+1) = \int_0^{+\infty} \int_0^{+\infty} (e^{-t-s})(e^{-t} + e^{-s} - e^{-t}e^{-s})^n \widehat{a}(t)\widehat{b}(s) dt ds.$$

Hence, if the numbers $C_n^{k,l}$ are defined by

$$(X + Y - XY)^n = \sum_{\substack{0 \leq k \leq n \\ 0 \leq l \leq n}} C_n^{k,l} X^k Y^l,$$

then, one has the following explicit formula

$$(a \bowtie b)(n+1) = \sum_{\substack{0 \leq k \leq n \\ 0 \leq l \leq n}} C_n^{k,l} a(k+1)b(l+1).$$

The name ‘‘harmonic product’’ is justified by the following harmonic property :

$$\left(\frac{1}{x} \bowtie a\right)(n) = \frac{1}{n} \left(\sum_{k=1}^n a(k)\right).$$

This harmonic property results from the equalities

$$\begin{aligned} \frac{1}{x} \bowtie a &= \int_0^{+\infty} e^{-xu} \left(\int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}} dt \right) du \\ &= \int_0^{+\infty} \left(\int_0^t e^{-xu} du \right) \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}} dt \\ &= \frac{1}{x} \int_0^{+\infty} (1 - e^{-xt}) \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}} dt \\ &= \frac{A(x)}{x} \quad \text{with } A(x) = \int_0^{+\infty} \frac{e^{-xt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt. \end{aligned}$$

Theorem 10. *Let*

$$\left(\frac{1}{x}\right)^{\bowtie k} := \underbrace{\frac{1}{x} \bowtie \frac{1}{x} \bowtie \cdots \bowtie \frac{1}{x}}_k \quad (k = 1, 2, \dots)$$

where $\frac{1}{x}$ denotes (improperly) the function $x \mapsto \frac{1}{x}$. Then, for all natural numbers m ,

$$\left(\frac{1}{x}\right)^{\bowtie(m+1)} = D\left(\frac{1}{x^{m+1}}\right) = \mathcal{L}\left(\frac{\Lambda^m}{m!}\right). \quad (27)$$

Proof. By (25) we have

$$D\left(\frac{1}{x^{m+1}}\right) = D\left(\underbrace{\frac{1}{x} \cdots \frac{1}{x}}_{m+1}\right) = \left(D\left(\frac{1}{x}\right)\right)^{\bowtie(m+1)} = \left(\frac{1}{x}\right)^{\bowtie(m+1)} \quad \text{since } D\left(\frac{1}{x}\right) = \frac{1}{x}.$$

Thus, (27) results from (10). □

Example 7.

$$\frac{1}{x} \bowtie \frac{1}{x} = D\left(\frac{1}{x^2}\right) = \mathcal{L}(\Lambda) = \frac{H(x)}{x} \quad \text{with } H(x) := \psi(x+1) + \gamma,$$

ψ denoting the logarithmic derivative of Γ . In particular, for each integer $n \geq 1$

$$\left(\frac{1}{x} \bowtie \frac{1}{x}\right)(n) = \frac{H(n)}{n} = \frac{H_n}{n}.$$

Theorem 11. *for all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has*

$$F_0(s) = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n^s} \quad \text{and} \quad F_k(s) = \sum_{n \geq 1}^{\mathcal{R}} \left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s} \right)(n) \quad \text{for } k \geq 1. \quad (28)$$

Proof. By (13), (23), (26) and the invariance of $\frac{1}{x}$ by D , one may write

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} \left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s} \right)(n) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D \left(\left(\frac{1}{x}\right)^{\bowtie k} \right)(n) D \left(\frac{1}{x^s} \right)(n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left(\frac{1}{x}\right)^k(n) D \left(\frac{1}{x^s} \right)(n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^k} D \left(\frac{1}{x^s} \right)(n) = F_k(s). \end{aligned}$$

□

In particular, by (11) and (27), one deduces from (28) the following identity :

Corollary 4. *For each natural number k ,*

$$F_k(1) = \vartheta(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n \geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{n}. \quad (29)$$

Example 8.

$$\begin{aligned} \vartheta(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma, \\ \vartheta(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n}, \\ \vartheta(3) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^3} = \frac{1}{2} \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^2}{n} + \frac{1}{2} \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}. \end{aligned}$$

Remark 8. Comparing (20) (applied with $q = k+1$) with (29) above, one may observe a kind of duality between $F_k(1)$ and $F_0(k+1)$. This results from the fact that $D = D^{-1}$.

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